Fractals and chaos

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26 March 2023

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Foreword

The starting point was the work on a manuscript "Artificial Intelligence - Past - Present - Future" (Springer - Vieweg 2023), which showed that many successes achieved through Deep Learning concern precisely fractal structures (the human nervous system, its blood circulation, diseases such as epilepsy. Parkinson's, dementia, tumours, etc.). I was still familiar with the basic terms up to about the year 2000, but I have only marginally followed further developments. This development is now to be traced here in a generally understandable way.

Classical mathematics - embodied, for example, in analytical geometry and analysis - is not able to describe many forms that occur in nature: sandstorms, dust clouds, earthquakes, coastlines, the surface of a cauliflower, an electroencephalogram, the growth of a tumour, the surface and temporal development of a dune and its migration, these are all phenomena that are well known, but which cannot be captured by school mathematics.

If one looks for properties that appear in all these phenomena, one finds after some observation of nature and after reading the relevant literature that the principle of self-similarity plays a central role. The possibility to characterise many properties by a fractal dimension is a big step towards the mathematical description of fractal structures. And last but not least, the constant development of computers plays a major role. The current state of the art makes it possible to simulate many fractal structures and to investigate how the models used correspond to the phenomena of the real world.

It is very interesting to see that many important mathematicians of the 17th, 18th, 19th and 20th centuries contributed to the theory of fractals without explicitly mentioning this structure: Gottfried Wilhelm Leibniz (1646 - 1716), Georg Cantor (1845 - 1918), Karl Weierstrass (1815 - 1897), S. N. Bose (1894 - 1974), Albert Einstein (1879 - 1955), Emmy Nöther (1882 - 1935), David Hilbert (1862 - 1943) and many more. However, the development of fractals is explicitly linked to the names of Karl Weierstrass and Benoît Mandelbrot.

The books [**1**], [**24**] and [**25**] provide an excellent basis.

1 Introduction

In the last 100 years, all fields that determine today's level of science and technology have developed strongly; computer science, electronics, information technology, etc. have emerged, as have space travel and the applications of nuclear fission and fusion. These developments were accompanied by a constant expansion of the demands on mathematics, which also had to constantly face new problems.

One of these new mathematical fields, which developed due to the new problems, is summarised under the term "fractals". Its mathematical foundations go back to the German mathematician Felix Hausdorff (1868).
- 1942) and the French mathematician Benoît Mandelbrot (1924 - 2010).
back. They are based, among other things, on the fact that many phenomena in the world do not run in a straight line and smoothly, as represented above all by Euclidean geometry, but look somehow dishevelled and crooked. The mathematical foundations and concepts necessary for their investigation will be presented here, with the intention of understanding them in parallel with the development of artificial intelligence, since these two fields must work together in many places.

1.1 Felix Hausdorff

Felix Hausdorff (born on 8 November 1868 in Breslau; died on 26 January 1942 in Bonn) was a German mathematician. He is considered a co-founder of general topology and made significant contributions to measure theory, functional analysis and algebra. In addition to his profession, he also worked as a philosophical writer and man of letters under the pseudonym Paul Mongré. He was persecuted by the National Socialists and took his own life in 1942 to avoid being sent to a concentration camp. After his habilitation, Hausdorff wrote one thesis each on optics, on non-Euclidean geometry and on hypercomplex number systems, as well as two papers on probability theory. His main field of work, however, soon became set theory, especially the theory of ordered sets. In the beginning it was a philosophical interest that led him to led him to study Georg Cantor's work in 1897. Hausdorff gave a lecture on set theory in the summer semester of 1901. This was one of the very first lectures on set theory, only Ernst Zermelo's college in Göttingen in the winter semester of 1900/1901 came into being a little earlier.

Figure 1.1: Felix Hausdorff (1868 - 1942)

Cantor himself never read about set theory. In the summer semester of 1910, Hausdorff was appointed as a scheduled associate professor at the University of Bonn. In Bonn he began with a lecture on set theory, which he repeated in the summer semester of 1912, substantially revised and expanded. In the summer of 1912, work also began on his opus magnum, the book "Grundzüge der Mengenlehre", which appeared in April 1914.

Hausdorff was appointed full professor at the University of Greifswald in the summer semester of 1913. This university was the smallest of the Prussian universities. The mathematical institute was also small; in the summer semester of 1916 and the winter semester of 1916/17, Hausdorff was the only mathematician in Greifswald. This meant that his teaching load was almost completely taken up by the basic lectures. It meant a significant improvement in his scientific situation that Hausdorff was called back to Bonn in 1921. Here he was able to develop a thematically wide-ranging teaching activity and repeatedly report on the latest developments.

research. Particularly noteworthy, for example, is a lecture on probability theory from the summer semester of 1923, in which he founded this theory axiomatically - measure-theoretically, and this ten years before A. N. Kolmo- goroff's "Grundbegriffe der Wahrscheinlichkeitsrechnung"". In Bonn, Hausdorff had outstanding mathematicians as colleagues and friends in Eduard Study and later Otto Toeplitz.

Anti-Semitism became state doctrine when the National Socialists came to power. Hausdorff was initially not directly affected by the "Law for the Restoration of the Professional Civil Service" passed in 1933, as he had already been a German civil servant before 1914. However, he was probably not spared the fact that one of his lectures was disrupted by National Socialist student functionaries. Thus he broke off his lecture "Infinitesimal Calculus III" from the winter semester 1934/35 on 20 November. Since a working conference of the National Socialist German Student Association was taking place at Bonn University during these days, which determined that the focus of the work in the current semester would be the topic of "Race and Nationality", it is very likely that Hausdorff's interruption of the lecture was connected to this event, since he never interrupted a lecture in his long career as a university lecturer.

On 31 March 1935, Hausdorff was finally given regular emeritus status after some back and forth. Those responsible at the time did not find a word of thanks for 40 years of successful work in German higher education. He continued to work tirelessly and, in addition to the expanded new edition of his set theory, published seven papers on topology and descriptive set theory, all of which appeared in Polish journals.

Hausdorff's estate also shows that he constantly worked mathematically in the increasingly difficult times and sought to follow current developments in the fields that interested him. Erich Bessel-Hagen (1848 - 1946) supported him selflessly by not only remaining loyal to the Hausdorff family in friendship, but also by providing books and journals from the institute's library, which Hausdorff, as a Jew, was no longer allowed to enter.

In 1939, Hausdorff tried in vain to obtain a research grant in the USA through the mathematician Richard Courant in order to be able to emigrate after all. Finally, in mid-1941, the deportation of the Bonn Jews to the monastery "Zur ewigen Anbetung" in Bonn-Endenich began. From there, they were later transported to the extermination camps in the East. After Felix Hausdorff, his wife and his wife's sister Edith Pappenheim, who lived with them, received the order to move to the Endenich camp in January 1942, they died together on 26 January 1942 by taking an overdose of Veronal.

Their final resting place is in the Poppelsdorf cemetery in Bonn. Between the time he was ordered into temporary storage and his suicide, he handed over his handwritten estate to the Egyptologist and presbyter Hans Bonnet (1887 - 1972), who was able to save it as far as possible despite the destruction of his house by a bomb hit.

Some Bonn Jews may still have had illusions about the Endenich collection camp - Hausdorff himself did not. E. Neuenschwander also discovered in the Bessel-Hagen estate the farewell letter that Hausdorff had written to the Jewish lawyer Hans Wollstein; here is the beginning and end of the letter:

"Dear friend Wollstein! By the time you receive these lines, the three of us will have solved the problem in a different way - the way you have constantly tried to dissuade us from. The feeling of security that you predicted for us once we had overcome the difficulties of the move does not want to happen at all, on the contrary! What has happened against the Jews in recent months arouses well-founded fear that we will no longer be allowed to experience a condition that is bearable for us."

After thanking friends and expressing last wishes regarding burial and will with great composure, Hausdorff continues writing:

"Forgive us for causing you trouble beyond death; I am convinced that you are doing what you can (and which may not be very much). Forgive us also for our desertion! We wish you and all our friends to experience even better times. Your faithfully devoted Felix Hausdorff".

This last written wish of Hausdorff's was not fulfilled: Attorney Wollstein was murdered in Auschwitz.

As early as 1918, he defined the Hausdorff dimension named after him, a measure for the roughness or, more precisely, for the fractal dimension of an object. For example, the Hausdorff dimension of a single point equals zero, of a line segment equals 1, of a square equals 2 and of a cube equals
That is, for point sets defining a smooth shape or a shape with a small number of vertices - the shapes of traditional geometry and science - the Hausdorff dimension is an integer consistent with the usual meaning of dimension, also known as topological dimension.ndexDimension!topological dimension However, formulas have also been developed that allow the calculation of the dimension of other, less simple objects, where one concludes, based solely on their properties of scaling and self-similarity, that certain objects do not have an integer dimension.

Figure 1.2: The grave of Felix Hausdorff at the Poppelsdorfer Cemetery in Bonn

have Hausdorff dimensions. Due to the significant technical advances of Abram Samoilovitch Besicovitch (1891 - 1970), which enabled the calculation of dimensions for highly irregular or "coarse" sets, this dimension is commonly referred to as the Hausdorff - Besicovitch dimension.

1.2 Benoît Mandelbrot

The French-US mathematician and non-fiction author is the father of fractal geometry. As an IBM Fellow at the Thomas J. Watson Research Center, Sterling Professor of Mathematics at Yale University and a staff member at the Pacific Northwest National Laboratory, he became a pioneer of theoretical physics and financial mathematics. The son of a Lithuanian-Jewish clothes merchant and a Polish doctor, he lived in France as a student during the German occupation and acquired his knowledge of mathematics and geometry mainly through self-study. After studies and research in Paris, Benoît Mandel- brot developed his theory of *fractal geometry* as an employee of IBM in the USA during the 1970s and 1980s; and coined the term "fractal".

Through the discovery of a geometric structure rich in form, the so-called Mandelbrot set, and the proof of fractals in nature, the mathematician succeeded in co-founding a completely new scientific discipline, chaos theory. These structures or patterns generally do not have an integer Hausdorff dimension, but a fractional value and also exhibit a high degree of scale invariance or self-similarity. This is the case, for example, when an object consists of several reduced copies of itself. Geometric objects of this kind differ in essential aspects from ordinary smooth figures.

Fractals are geometric shapes or forms represented in natural objects, from a fern leaf to a spider web or snowflake to larger phenomena such as clouds, hurricanes or even galaxies in space. The discovery of fractals made it possible to develop irregular shapes that until then could not be defined or represented by mathematics. Classical geometry had been dealing with squares, rectangles, cubes, spheres, etc. for thousands of years, but there was no way to describe clouds, treetops, coastlines, river courses and their deltas mathematically. One can already see in Fig.
1.5 very nice that the edges of the snowflake parts consist of straight lines, but the image as a whole starts to fill the plane.

Figure 1.3: Benoît Mandelbrot (1868 - 1942)

Mandelbrot himself started with the problem of determining the length of the coastline of Great Britain: *How long is the Coast of Britain?* One proceeds in the following way: a point on the coast, for example Brighton on the south coast, is fixed. From there, use the straight-line distance of 10 km to the left and to the right. You continue this procedure around the whole island until you meet somewhere in Scotland. The end points will not meet exactly, but the sum of all these lines of 10 km is an approximation for the length of the coast. Bays or capes are ignored, it is assumed that they are essentially the same. If you have enough time, you can reduce the scale to about 1 km. The length is now determined more precisely, its value will be larger. So the real length depends on the scale. A relatively exact value would be obtained by walking the entire coastline, which of course would encounter difficulties in some places. Today, satellite images and image processing with computer programmes are used. A very common fractal shape, found both in fractals and in nature as a whole, is the spiral . This is one of the most common shapes represented as a fractal; it is often seen on the internet. This mathematical shape is also common throughout nature.

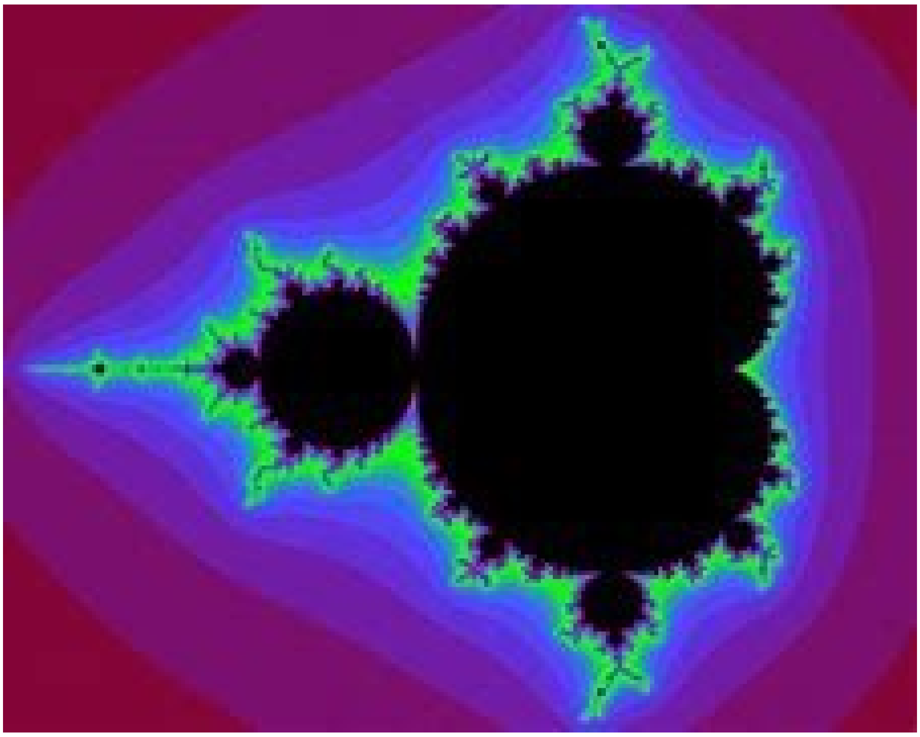


Figure 1.4: The Mandelbrot set

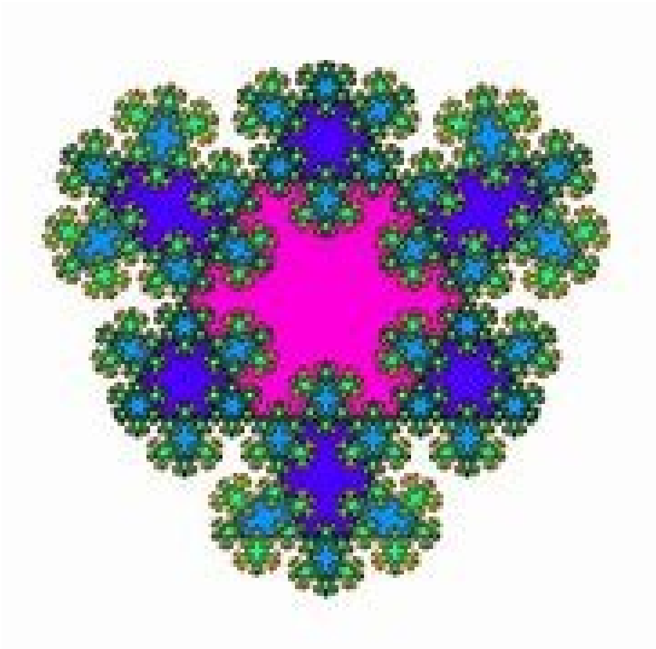


Figure 1.5: A variant of a Koch snowflake

It is described by the equation

$$T_w(x) = \sum_{n=0}^{\infty} w^n \cdot s(2^n x)$$

with a parameter w and $s(x) = \min_{n \in \mathbb{Z}} |x - n|$ (the distance from x to the nearest integer). The functions $T_w(x)$ are continuous but not differentiable in the interval $[0, 1]$.

1.3 Self-similarity

A strict self-similarity exists if a change in the scale used leads to a similar object. For example, if there is a circle with the radius r, then the following equations apply to the circumference U and the area F

$$U = 2 \cdot r \cdot \pi \tag{1.1}$$

$$F = \pi \cdot r^2. \tag{1.2}$$

If you double the radius and use $r' = 2r$, the same formula applies to U' and F', you just have to replace r with r'. You can even include a point by using $r' = 0$. If one uses the same centre point for all circles, then they are concentric circles.

For straight lines, one can use compression as a means of creating self-similar figures. In a Cartesian coordinate system, one uses for a straight line the equation

$$y = mx + n. \tag{1.3}$$

If $m > 1$, then all points move further to the right, for $m < 1$ they move to the left. For $n = 0$, the straight line goes through the zero point, for positive n it is shifted upwards, for negative it is shifted downwards. So there are two parameters whose change produces strictly similar figures.

Spheres are defined by the equation

$$x^2 + y^2 + z^2 = r^2 \tag{1.4}$$

defined. Here again, self-similar figures can be produced by changing the radius. For $r = 0$, one can even regard the zero point as a sphere.

All in all, it can be said that all one-, two- or three-dimensional figures described by suitable equations of analytic geometry can be the starting point of strictly self-similar figures. Figure 1.6 shows a spiral that is described by the equation

$$r = 2 - 2 \sin \theta$$

is generated. For $\theta = \pi/2$, $\sin \theta = 1$, so $r = 0$. For $\theta = 3\pi/2$, $\sin \theta = -1$, so $r = 4$. Due to the periodicity of the sinusoidal function, it then returns to zero and runs through the curve again and again. One can now replace the constant 2 with another value, as well as the factor 2, and variants $\theta\ \alpha$ are also possible for θ, where α can be any real number. With this, one can again create a large set of self-similar curves.

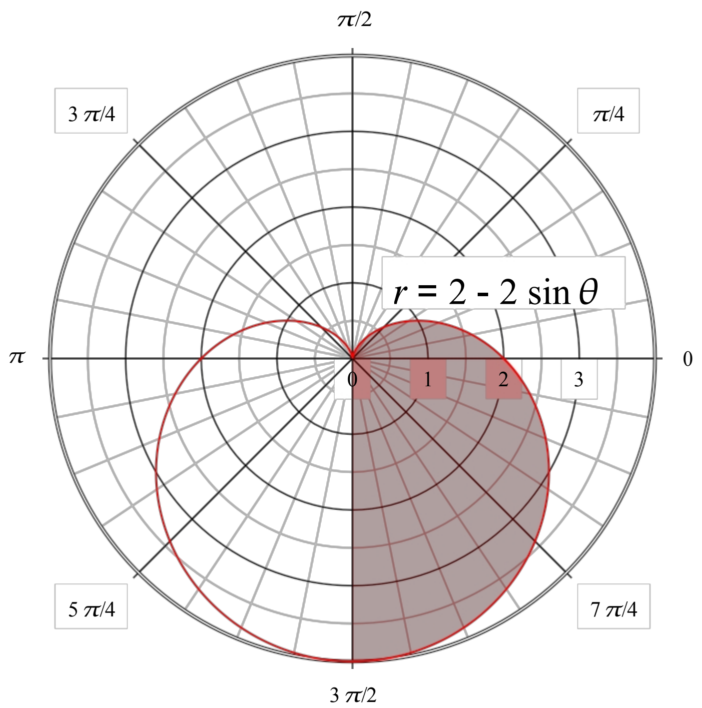


Figure 1.6: Spiral

Spirals can be found in many areas of nature and technology. The sky shows us the most beautiful spiral forms. They are formations of billions of suns in which cosmic matter is arranged in a spiral. Only recently (in the year 2000) it was discovered that there is a black hole in the centre of every galaxy that creates the spiral vortex! According to the latest findings, the black hole is the centre of every galaxy and creates the spiral shape. It is both a devouring maw (chaos) and a star-former (order).[**29**]

Even the smallest units of matter, subatomic particles with an electric charge, move along spiral paths when a magnetic field deflects them.

1
Introduction

The spiral shape is a result of adaptations to living conditions: Butterflies, for example, have long proboscises to suck nectar from the calyxes. In some tropical species, these proboscises can measure 25 centimetres in adaptation to long-tubed flowers. During flight, such a long organ would of course be a hindrance. Rolling it up into a spiral is the easiest way to deal with the problem. In many cases, the behaviour of a living creature also leads to spirals: For the caddis fly, which lays its tiny spherical eggs on the underside of floating leaves, the spiral was the most economical way to do this.

In the realm of plants, the spiral shows itself as a form that unfolds to become what is to become. Spiral growth has great functional advantages here too: In buds, a large number of flowers and leaves can be held in readiness in a very small space, wrapped protectively several times over, and unfolded with a tiny twist. Many plants grow spirally upwards towards the sun, and many leaves arrange themselves in spirals to make the best use of the light. Trees such as spruces reveal a spiral growth when studied closely - from the roots all the way up to the treetop

The fossilised spiral-shaped shells of the long-extinct ammonites and the likewise fossilised smallest shell-bearing animals, the tiny one-cell foraminifera, still reveal growth processes and their functional sense after millions of years.

Despite their great variety, snail shells are constructed according to strict geometric laws.

Probably no other living being shows more clearly that shape grows into a genetically predetermined form. Every cell knows at all times about the whole, about the idea that makes it grow. Thus, according to species-specific blueprints, different shapes emerge, all of high precision and beauty.

Most snail shells are right-twisted, only very few species are left-twisted. Very rarely do wrongly-wound, mirror-image symmetrical specimens occur within a species, so-called inverses - in the case of the right-wound Weinberg snail, one in 5000. The fact that mirror-image counterparts of a snail shell exist extremely rarely is an expression of a fundamental difference between animate and inanimate nature. First of all, it could be assumed that right- and left-handed shapes are statistically equal.

However, living nature prefers one of the two possibilities: The molecules of sugars always appear in the right-hand form, while those of amino acids always appear in the left-hand form - provided they are involved in life processes. The mirror image

However, equality is also violated in inanimate nature: the decay scheme of particles produced in the radioactive beta decay of atomic nuclei also proves to be asymmetrical.

The cochlea may be regarded as a conch shell with a reversed function. It is divided longitudinally by the basilar membrane, on which the actual sensory organ, the Gortic organ (organon spirale), lies. There, hair cells act as receptors and convert the vibrations of the sound waves into electrical impulses, which are then conducted to the brain.

Whirlwinds and water vortices run counterclockwise in the northern hemisphere, but with it in the southern hemisphere. This makes it possible to locate the equator exactly. No movement of water occurs there at all.

1.4 Fractals on the chessboard

One also finds fractal structures in places where it is not to be expected.

The queen problem is a chess mathematical problem. Eight queens are to be placed on a chessboard in such a way that no two queens can capture each other according to their moves as defined in the rules of chess. The colour of the pieces is ignored and it is assumed that any piece could attack any other piece. Pieces arranged in this way on the chessboard are also called independent. For queens, this means, in concrete and different terms, that no two queens may stand on the same row, line or diagonal.

In the centre of the queen problem is the question of the number of possible solutions. In the case of the classic 8-8 chessboard, there are 92 different ways of placing the queens. If we consider solutions to be the same, which result from mirroring or rotating the board apart, 12 basic solutions remain.

The problem can be generalised to square chessboards of any size: Then the task is to position n independent queens on a board of n n squares (with the natural number n as parameter).

The smallest value with a solution of the problem is $n = 4$. There exists only one essential solution, the second solution can be obtained from this first solution by

Create a reflection on the central horizontal or vertical. This is shown in the following diagram:

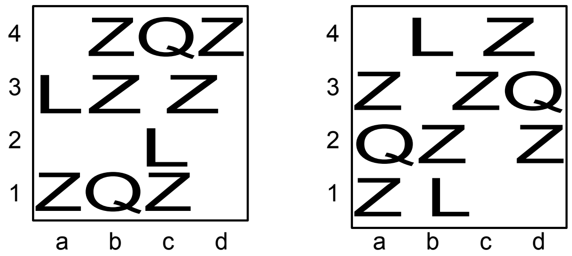


Figure 1.7: All solutions for four queens on a 4 × 4 chessboard To construct a

fractal structure, one can use n = 5.

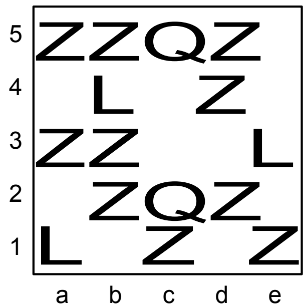


Figure 1.8: A solution for five queens on a 5 × 5 board

Now replace each square of this board with the solution of board 5 5; this results in a board 25 25. The solution of board 5 5 is now entered in the second board of the first column; this means that on board 25 25 the rows 1 to 5 and the columns 1 to 5 are used. This continues until all rows and columns contain exactly one queen.

The result follows the rule that there is exactly one queen in each row and in each column, the whole board is covered for some reason, but that there is a safety distance between the players. This procedure can be applied, for example, to rules for fighting an epidemic.

In the book by Manfred Schröder [**1**], the author starts with a board 5 x 5 and shows its fractal continuation. All the representations for 5 x 5 can be continued again to 25 x 25, 125 x 125 etc. There is also a lot of room for manoeuvre here.

We will always encounter the properties of horizontal and vertical mirroring as well as mirroring at a central point in fractals. In many interpretations and applications, it is not only the fact of a safety distance between the occupied positions that is important, but also the fact that every line and each column contains exactly one element. Curiously, the solution for 5 × 5 still has the property that the five marked squares allow a knight to make a round trip using only marked squares.

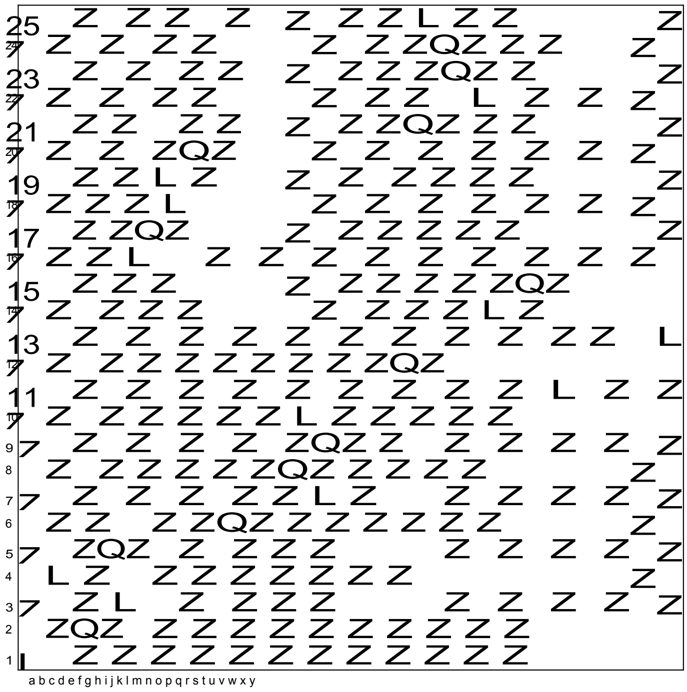


Figure 1.9: A solution of the checker problem on a board 25 x 25

Many other solutions can now be constructed from this solution: for example, one replaces each field with this solution 25 x 25 and thus obtains a solution for a board of size 625 x 625. Solutions for smaller boards are obtained if one selects an element and deletes the associated column and row at the same time.

1.5 Affine Images

The concepts of mirroring, shifting, etc. that appeared in the previous section are special cases of affine mappings. They will now be presented in general.

A mapping is generally a function f that maps a space **X** to a space **Y**, formally f : **X** → **Y** , where each point of the space **X** is mapped to a point of the space **Y.** An affine mapping **w** in the plane thus has the general form

$$\mathbf{w} : \mathbb{R}^2 \rightarrow \mathbb{R}^2. \tag{1.5}$$

The affine mappings include translation, scaling, rotation, reflection and transvection (shear). All these mappings and their combinations can be defined by a matrix A and a displacement vector

$\vec{v}$ represent:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} e \\ f \end{pmatrix}$$

In summary, an affine mapping w can be described by the formula

$$w : \vec{p}\,' = \tag{1.6}$$

$$w : \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = A\vec{p} + \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \tag{1.7}$$

describe.

In order to calculate the coordinates of a point (x,y), one must first perform the matrix multiplication

$$A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a \cdot p_x + b \cdot p_y \\ c \cdot p_x + d \cdot p_y \end{pmatrix} \tag{1.8}$$

carry out.

Then one adds $\vec{v}$ and gets

$$A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a \cdot p_x + b \cdot p_y + e \\ c \cdot p_x + d \cdot p_y + f \end{pmatrix} \tag{1.9}$$

for the two coordinates of the image point.

Various affine mappings now exist, all of which produce self-similar structures.

1. Translation um einen Vektor

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

If one uses the unit matrix for A, then the original coordinates are retained.

$$wt: \begin{pmatrix} p'x \\ p'y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad (1.10)$$

2. Change of size (scaling)

Here you change the size of an object in the x- and the y-direction. If the change factors are the same, then the original shape remains, if they are different, then a square can become a rectangle, for example, or vice versa.

$$wt: \begin{pmatrix} p'x \\ p'y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad (1.11)$$

Here a and b are supposed to be positive numbers.

3. Counterclockwise rotation around the zero point with a subsequent displacement

$$wt: \begin{pmatrix} p'x \\ p'y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad (1.12)$$

4. Contraction: This is also a very interesting form of mapping. It can be applied if one can define a distance $d(x, y)$ between two elements. If two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are given in the plane, their distance is

$$d(x, y) = +\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If one now has a figure λ in which the distance always decreases by a factor $\lambda < 1$, then the figure becomes smaller and smaller and finally contracts to the zero point. All figures in between are self-similar.

The different types of affine mappings can also be effortlessly coupled with each other. The drawing below shows a shift followed by a rotation.

The next drawing is by Hugues Vermeiren and also shows the combination of different illustrations.

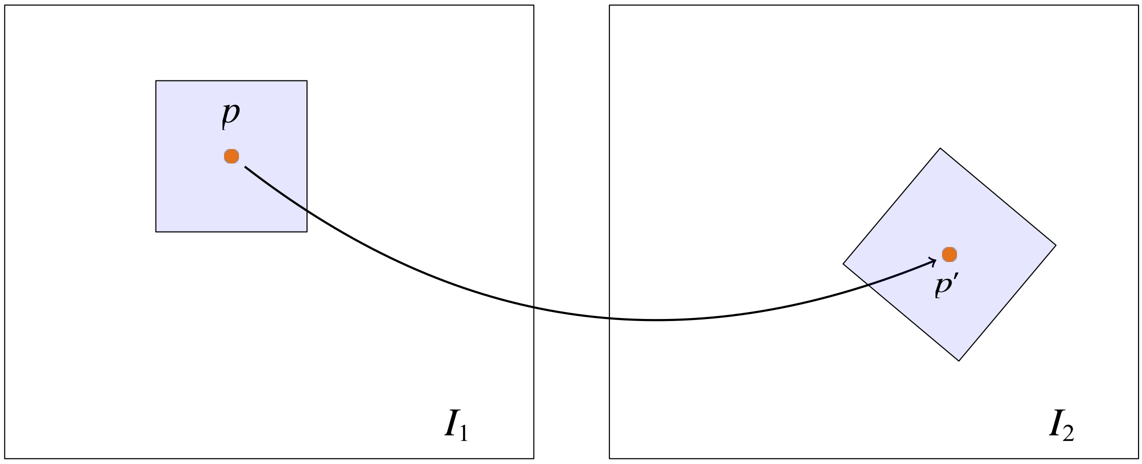


Figure 1.10: Displacement and rotation

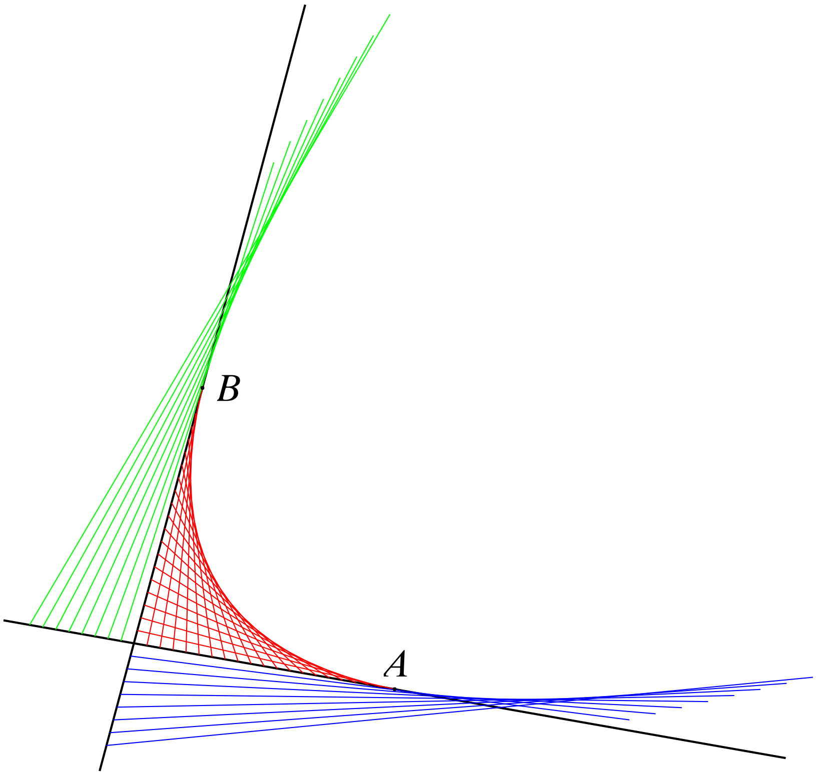


Figure 1.11: Construction by Hugues Vermeiren

2 Basics

2.1 Forerunner

The earliest fractals are by `D. Hilbert` and `G. Peano`, without being called fractals when they were first described. The `Hilbert curve` is a `mirror-symmetrical` and `space-filling` curve in the two-dimensional plane. It can easily be generalised to higher dimensions.

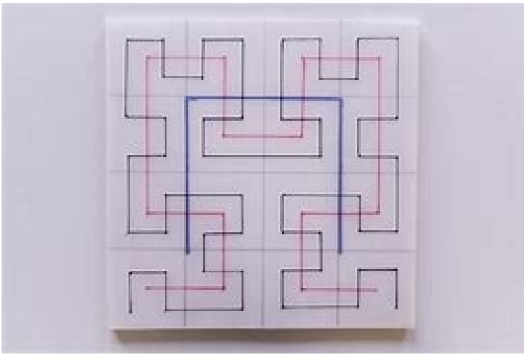


Figure 2.1: A two-dimensional Hilbert curve

The `Peano curve` is a `point-symmetrical` and `space-filling curve` in the two-dimensional plane. It can easily be generalised to higher dimensions. One starts by dividing a square into nine squares of equal size, which are traversed in an S-curve. In the next step, each of these squares is subdivided again and the resulting squares are traversed again in S-curves, which are joined together as a new curve, and so on.

D. Hilbert and G. Peano contributed a great deal to building mathematics on secure foundations. In the process, they advanced as far as fractals, without this term being used at the time.

The `Sierpinski triangle` is defined with the help of self-similar triangles in the two-dimensional plane, which can be equilateral triangles or general triangles. The three-dimensional variant is the `Sierpinski tetrahedron`.

The Sierpinski triangle can also be extended to three dimensions, resulting in `Sierpinski simplexes`. Because of their allegedly very strange properties, fractal curves were also called `monster curves in the past.`

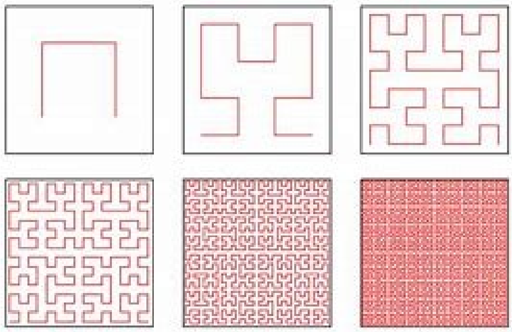


Figure 2.2: A Peano curve

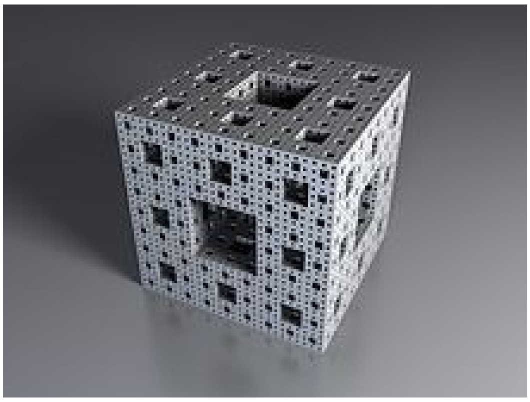


Figure 2.3: A Menger sponge

Fractal patterns are often generated by recursive operations. Even simple generation rules result in complex patterns after a few `recursion steps.`

Fig. 2.4 shows a `Pythagorean tree.` A square is constructed on a base line. On this basic element (trunk) a Thales circle is drawn on the upper side and this is divided as desired. The resulting point is connected to the base element so that a right-angled isosceles triangle is created. A square is constructed from the two sides of the triangle, a Thales circle is drawn, this is divided, a right-angled triangle is constructed and thus extended to form a square again. This process is repeated as often as desired.

In the following, let the side length of the first square (the "stem") be equal to 1. If the interior angles of the firstre√xed triangle are equal to 45°, 45°, and 90°, the Side lengths therefore equal to 2/2, 2/2 and 1, the Pythagorean tree is symmetric. The axis of symmetry is the median perpendicular of the hypotenuse of the first right-angled and isosceles triangle.

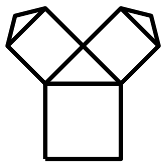


Figure 2.4: The first steps in the construction of the Pythagorean tree

The initial square has a height of 1. The triangle above this square has a height of 1 2. Thus the height to the middle corner of the smaller squares is equal to 2. The total height h, which can be obtained by continuing this procedure, is the sum of an infinite series

$$h = 2(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = 4. \qquad (2.1)$$

You can increase the height of the tree as much as you like if you do not choose $a = 1$, but a larger value. Then the height is $h = 4 - a$. For the width we get $b = 6 - a$. A complete circumnavigation of the tree results in a length of ≈ 25, $313 - a$.

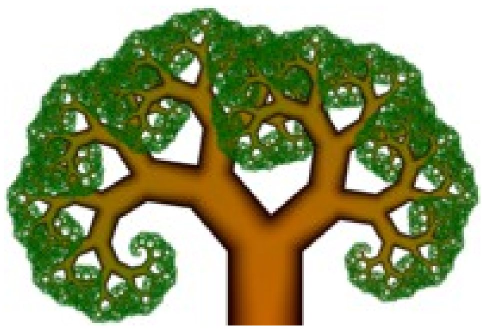


Figure 2.5: A Pythagorean tree

The appearance of the Pythagorean trees can be modified in many ways. By using different colours for squares and triangles, one can create very beautiful pictures. From the website https://www.redusoft.de/info/download- demo.html you can download a very nice demo version of the programme MathProf 5.0. There you can find many other interesting programmes for mathematics and physics.

You can see here again the effectiveness of the biological construction: on an area of $24 - a^2$ you achieve 25 times the biologically active surface.

Another fractal is the Newton fractal. It maps the set of complex numbers in itself, i.e. the variables and the function values are generally complex numbers. It is represented by the function

$$f_z \to f(z) = z - \frac{p(z)}{p'(z)} \tag{2.2}$$

and is used to find the zeros of the function p:

$$f(z) = z^3 - 2z + 2, \tag{2.3}$$

which describes the Newton method for finding zeros of the function z^3 $2z+2$. Depending on the starting point z_0 , the sequence can be

$$z, f(z), f^2(z), f^3(z), \ldots$$

show quite different behaviour. It must be noted that $f^n(z)$ means the n-fold application of the function $f(z)$.

The procedure starts with a value z_0 and calculates the value $z_{k+1} = f(z_k)$. The colours express the convergence speed and the three zeros. Start values that lie in the beige areas converge to the same zero (in light beige in Fig. 2.6), analogously for the green and blue areas. The zeros for the green and blue areas lie symmetrically to the horizontal axis of symmetry on the right (Fig. 2.6). The faster a starting value converges to its zero, the lighter it is coloured. The values in the red areas do not converge towards a zero, but are captured by the attracting cycle {0,1}. The Newton fractal - recognisable in the image as a bright structure - is not limited.

In the three directions that can be recognised, it reaches ∞.

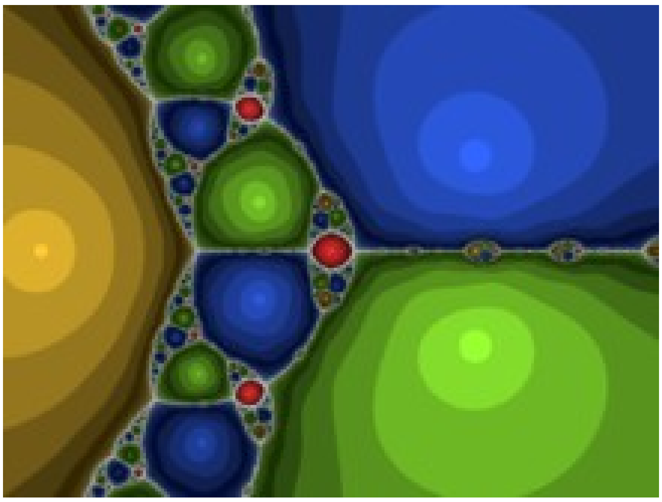


Figure 2.6: Newton fractal for the function $z^3 - 2z + 2$

Since you can apply the Newton method to any complex function, you can also create lots of beautiful fractals. The colouring comes from changing the colour after a certain number of steps. If you choose the colour sequence from light to dark, you can see directly in which way you are approaching a zero.

Figure 2.7: A three-dimensional representation of Covid-19 - viruses

Many problems in science and technology have been successfully solved in the past. Classical mathematics assumes that all objects are exactly (sharply) defined and that statements are always formulated so precisely that they reflect a fact exactly or miss it, i.e. that every statement is either true or false. Parabolas, for example, describe the trajectory of a body that is thrown upwards at an angle. The orbits of comets can also be approximated as parabolas when they fly past the Earth. Similar applications exist for ellipses and hyperbolas.

However, most curves and surfaces of our environment cannot be described exactly by equations of analytical geometry. The surface of a cauliflower, the edges of broken porcelain, cobwebs and many more are some examples.

2.2 The Cantor- crowd

The Cantor set, also called Cantor's discontinuum, Cantor dust or wipe set, is a certain subset of the set of real numbers with special topological, measure- and set-theoretical as well as geome- trical properties. It is named after the mathematician Georg Cantor (1845 - 1918), to whom general set theory essentially goes back.

The construction of the Cantor set is a simple way to construct fractal structures:

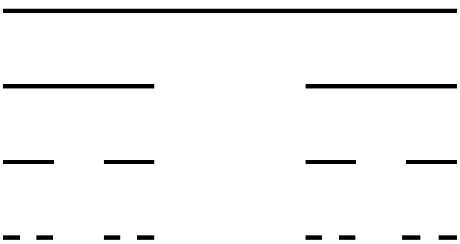


Figure 2.8: Cantor's dust

Start with a line of length 1 and remove the middle third; you get the closed intervals $[0, 1/3]$ and $[2/3, 1]$, the points of the open interval $]1/3, 2/3[$ were removed. Proceed in the same way with the remaining distances, always removing the middle third. The lengths of the distances are
Row after

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$$

The number of elements at each level is growing exponentially, the distances are getting shorter and shorter, and they are getting closer and closer together. The expression "dust" has its justification. It doesn't matter on which level you are, every point is separated from its neighbours to the left and right. Nevertheless, the dust contains just as many elements as the original interval $[0, 1]$. The sections that were eliminated in the construction process did not result in fewer elements. This is an essential property of sets that possess the power of the continuum (the set of real numbers). Supra-countable subsets of the set of real numbers are equally powerful.

Consider the two closed intervals [0, 1] and [0, 10]. The figure

$$y = 10 \cdot x \tag{2.4}$$

$$x = y/10 \tag{2.5}$$

assigns exactly one element of the other interval to each element of the one interval. So these two intervals must be equally powerful. The power of the set of natural numbers has the same power as that of the rational numbers and is denoted by $\aleph_0$, that of the real numbers $\aleph_1$. $\aleph$ is the first letter of the Hebrew alphabet.

The Hausdorff dimension of the Cantor set is equal to $\ln 2/\ln 3 \approx 0,6309$; it is therefore less than 1, but it is greater than 0. The dust is therefore a middle thing between a point and a one-dimensional curve (Fig. 2.8).

Figure 2.9: A sandstorm - Cantor dust in reality

The Cantor set is compact, closed, has no isolated points and is totally disjoint (a discontinuum) and nowhere dense. Its area is equal to zero. It is self-similar and has a non-integer Hausdorff dimension; its thickness is equal to $\aleph_1$.

The approach that led to the Cantor set was based on simply eliminating the middle part.

The procedure for constructing the Cantor dust can be modified even further. At each iteration, the length of the left interval is defined by a random variable C_1 , which defines the length of a subinterval of the original segment. The same applies to the right interval with a random variable C_2 . The Hausdorff dimension s then satisfies the equation $E(C_1^s + C_2^s) = 1$. E(X) is the er-maintenance value of X. In the literature, a Hausdorff dimension of 0.7499 is given for $E(C_1) = 0,5$ and $E(C_2) = 0,3$.

For the constructions of sets similar to the Cantor dust, one can also use powers of rational numbers, for example the sequence

$$(1/2)^n = (1, 1/2, 1/4, 1/8, \ldots) .$$

You go from $x = 1$ to the left and omit every second interval, first the one between 1/2 and 1/4, then the one between 1/8 and 1/16 and so on. The omitted intervals become smaller and smaller. It is interesting if powers of 1/3 are used next. The elements of the set constructed with this also push more and more towards the zero point, but much faster.

Similar structures also occur in space.

Fig. 2.10 shows the Saturn Nebula. At its equator, one can see concentrated

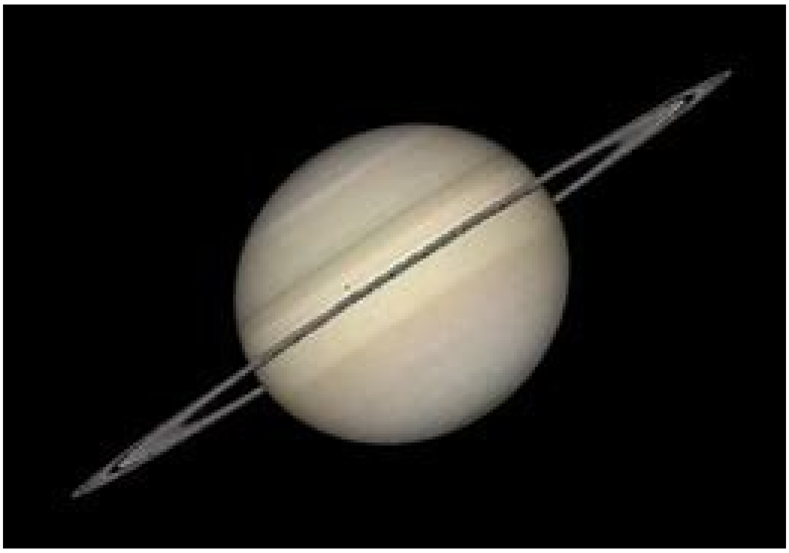


Figure 2.10: Saturn and its rings

circles that consist of a kind of dust. They are close together, but the particles do not seem to move from one circle to another.

2.3 The Hausdorff - Dimension

At the end of the previous section, a Haus Dorff dimension of 0.6309 was given for the Cantor dust. Now we will describe how these fractional dimensions come about.

For a set of points of finite extent in a three-dimensional space, consider the minimum number n of spheres of radius r required to cover the set of points. This minimum number is a function $n(r)$ of the radius r. The smaller the radius, the larger n. From the power of r, with which $n(r)$ approaches zero for the limit r, the Hausdorff dimension is calculated as follows D:

$$n(r) \sim \frac{1}{r^D}. \tag{2.6}$$

The solution of this equation then results in

$$D = -\lim_{r \to 0} \frac{\log n}{\log r}. \tag{2.7}$$

Instead of spheres, cubes or comparable objects can be used just as well. For point sets in the plane, circles can also be used for covering.

can be used. For point sets in more than three dimensions, higher-dimensional spheres must be used accordingly.

2.4 The fig tree- Attractor

The equation used was originally introduced in 1837 by Pierre François Verhulst as a demographic mathematical model. The equation is an example of how complex, chaotic behaviour can arise from simple non-linear equations. The associated dynamics can be illustrated by a so-called fig tree diagram (Fig. 2.10) [**4**]. It is based on the following equation:

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n) = -r \cdot x_n^2 + r \cdot n. \quad (2.8)$$

x_0 is a value between 0 and 1, r is between 0 and 4. According to the value chosen for *r*, the value sequences of x behave differently. This behaviour does not depend on the starting value x_0 , but only on r:

- With *r* from 0 to 1, you always get 0 after a few iterations.
- With *r* between 1 and 3, the limit is $1 - \frac{1}{1-r}$ on.
- With *r* between 3 and $1+\sqrt{6} \approx 3,45$ the sequence changes at almost all starting values, except 0, 1 and $1 - \frac{1}{r}$, between two values, called attractors.
- With *r* between $1+\sqrt{6} \approx 3,45$ and $\approx 3,54$ the sequence alternates between four attractors at almost all starting values.
- If *r* is greater than 3.54, first 8, then 16, 32, etc., attractors are set. attractors set in. At $r \approx 3,57$ the chaos begins.

A fig tree diagram (Fig. 2.11) is created by plotting the pairs of values (r, x_n) One chooses a certain number of iterations n for the diagram, which is then maintained for all calculated points of the diagram. For each value *r*, a series of x_0 -start values between 0 and 1 is then chosen randomly or in order. For each starting value, the equation is calculated together with the chosen r x_n iterated - times. For the values x_n calculated in this way, a point is then drawn at *r* in the diagram. This is repeated for the entire range of *r*.

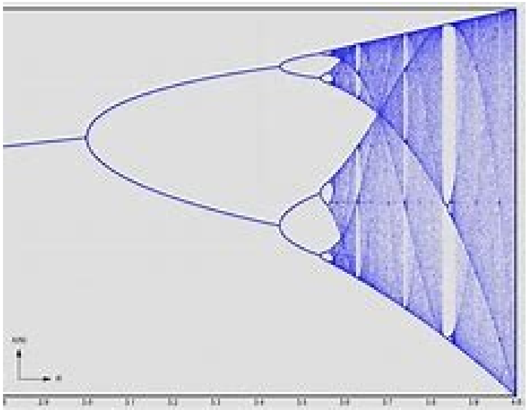


Figure 2.11: A fig tree - diagram

Each point in the diagram is the result of a fixed number of n iterations for a certain starting value x_0 and a certain parameter r. In the areas where single lines are visible, almost all start values converge towards the same end values x_n for a certain r-value. These values or lines are therefore called attractors. In the other areas, a different final value is obtained for each start value. However, these end values do not lie on a line, but are distributed (pseudo-) randomly.

2.5 The length of the Koch - curve

When constructing a Koch curve (Fig. 2.12), one starts with a line of length 1. This is divided into three parts. The middle part is omitted, instead two sections of length 1/3 are applied. The total length of the curve is now 4/3.

At each iteration step, each stretch of the curve is replaced by four stretches with 1 3 of the length of the curve (Fig. 2.13). The curve thus becomes longer by a factor of 4 3 with each iteration step. After the iteration step n, the length of the curve has thus increased to 4 3 ^{n} if the initial stretch had the length 1. Consequently, the length continues to increase as the procedure is continued. The area below the curve, on the other hand, is limited. It approaches the value 3 20 0, 0866 with increasing n. According to its construction rule, the Koch curve is strictly self-similar, i.e. the same structures appear again and again at any magnification. It has a Hausdorff dimension of

$$\frac{\log 4}{\log 3} \approx 1,262.$$

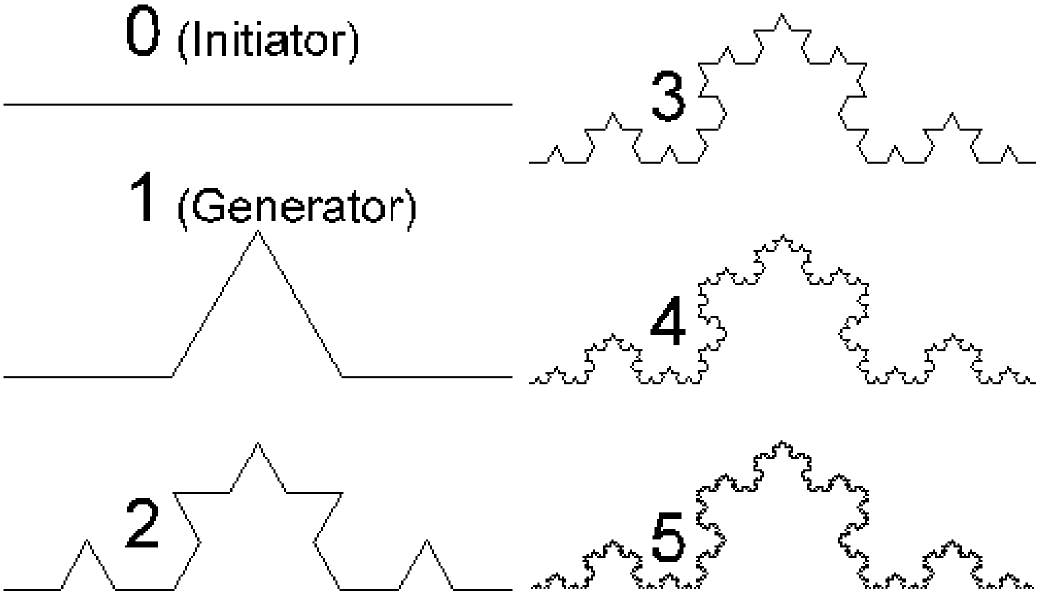


Figure 2.12: The construction of the cooking curve

So the curve gets longer and longer, but the area used remains the same. Many biological systems use this structure, it allows to increase the number of certain elements without increasing the volume or area used.

The Euclidean plane can be parquetised with Koch snowflakes of two different sizes. This parquetry is periodic, mirror symmetric, point symmetric, rotational symmetric and translation symmetric (Fig. 2.13).
Here, the dimensions of the yellow snowflakes are larger than those of the green snowflakes by a factor of $\sqrt{3}$ 1 , 73. The total area covered by yellow flakes is three times as large as the area covered by green spots.

The constructions of the Koch curve date from 1904. Helge von Koch was a Swedish mathematician (1870 - 1924). He constructed the Koch curve named after him, one of the first fractals, as an example of an infinitely long curve that cannot be differentiated at any point.

2.6 Sierpinski - Structures

The Sierpinski Triangle is a fractal described in 1915 by the Polish mathematician Wacław Sier- piński (1882 - 1969) - sometimes also called Sierpinski Surface or Sierpinski Triangle.

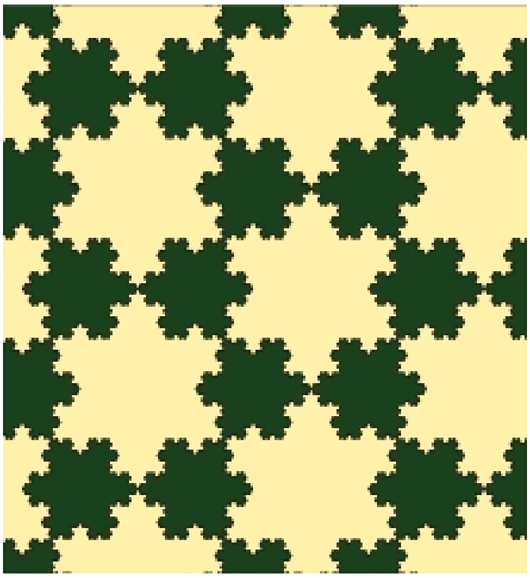


Figure 2.13: A parquet made of Koch's flakes

Figure 2.14: A 60-fold dodecahedron

This is called a gasket, which is a self-similar subset of a mostly equilateral triangle.

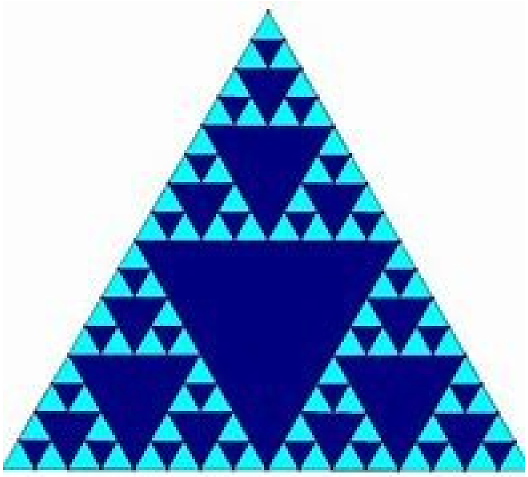


Figure 2.15: A Sierpinski triangle

In the large equilateral triangle, connect the centres of the three sides and cut out the middle dark triangle. Three smaller triangles, also equilateral, remain at the corners and are treated in the same way. More and more dark triangles are created, all of which lie within the initial triangle (Fig. 2.15). The area of the initial triangle is

$$F = \frac{\sqrt{3}}{4} \times a^2 \approx 0,433 \times a^2 . \tag{2.9}$$

The four triangles that are created - the dark one in the middle and the three shaded ones at the corners each have an area of

$$F' = \frac{\sqrt{3}}{16} \times a^2 \approx 0,108 \times a^2 . \tag{2.10}$$

Now subtract the area of the inner triangle from the total area and you get

$$F_1 = \frac{\sqrt{3}}{4} \times a^2 - \frac{\sqrt{3}}{16} \times a^2 \approx 0,325 \times a^2 . \tag{2.11}$$

If you now want to use the edges for a specific purpose, you now have the outer edges with the length $3a$ available and additionally three inner edges with the length $a/2$, which limit the dark triangle, i.e. a total length of $4,5\,a$. If one continues the procedure, the total length of the available edges becomes larger and larger without having to use a larger area.

The Sierpinski carpet is created when squares are used instead of triangles. The fractal dimension is ln 8/ln 3.

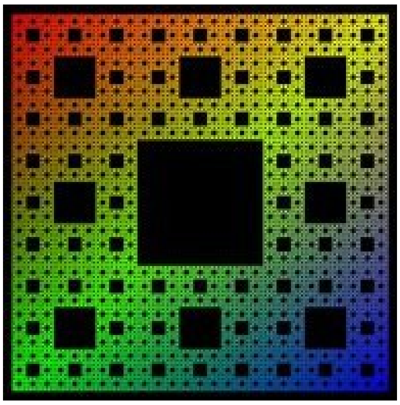


Figure 2.16: The Sierpinski Carpet

Figure 2.17: Three-dimensional flakes

One can also become a pyramid as a starting point. From a given large pyramid, smaller and smaller pyramids are removed, similar to what happens with the Sierpinski triangle.

Figure 2.18: A three-dimensional generalisation of the Sierpinski triangle

2.7 Fractals derived from surfaces and bodies Fractals

Many other geometric figures can serve as a starting point for the definition of fractals. This is very simple for circles and spheres (Fig. 2.19).

Figure 2.19: Fractal circles and spheres

Fig. 2.20 again shows the principle of forming a large surface area with a relatively small volume.

Figure 2.20: Interlocking curled spherical surfaces

In the case of the head and the brain, too, one can still guess at the spheres in the background, but they are further refined (Fig. 2.21).

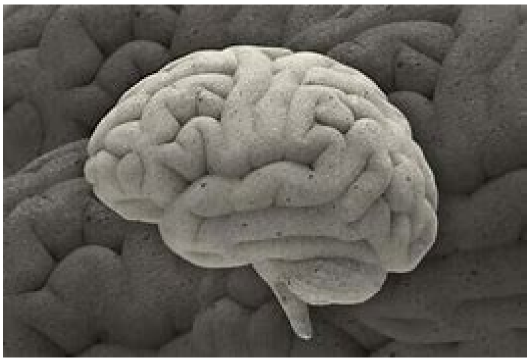


Figure 2.21: The fractal structure of the brain

The Rössler attractor (after Otto E. Rössler, * 1940) is an attractor defined by the following system of differential equations:

$$\frac{dx}{dt} = -(y + z) \quad (2.12)$$

$$\frac{dy}{dt} = x + ay \quad (2.13)$$

$$\frac{dz}{dt} = b + (x - c) - z. \quad (2.14)$$

According to Otto E. Rössler, this model was inspired by the observation of a taffy puller on Coney Island, which repeatedly stretches and folds its taffy mass. The Rössler attractor does not describe a real physical system. It is an academic construct that is intended to simply illustrate certain chaotic effects.

Fig. 2.22 shows very nicely that only equation 2.14 is non-linear due to the multiplication $x\,z$. In the $x\,y$-plane things are quite calm. The fractal dimension of the Rössler attractor is slightly larger than 2. For a = 0.1, b = 0.1 and c = 14, it is estimated to be between 2.01 and 2.02 (see Fig. 2.21). The value of the fractal dimension indicates that this is essentially a surface with a small detour into the third dimension.

2.8 Percolations

Percolation theory is one of the simplest models for disordered systems. It was developed to deal mathematically with disordered media in which the

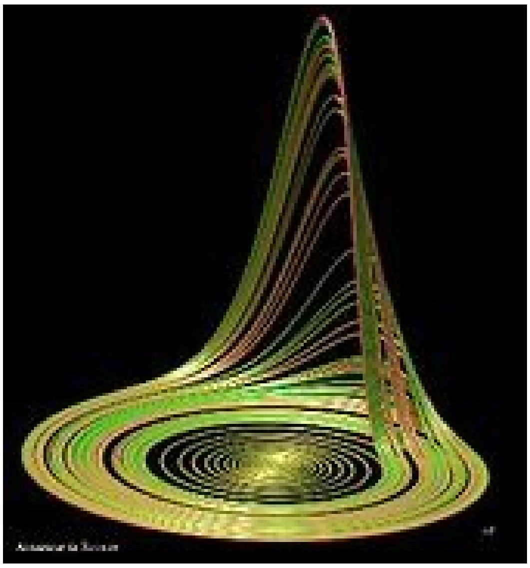


Figure 2.22: The Rössler attractor

ness is defined by a random variation of the degree of connectivity. The main concept of percolation theory is the existence of a percolation threshold above which the physical properties of the entire system change drastically.

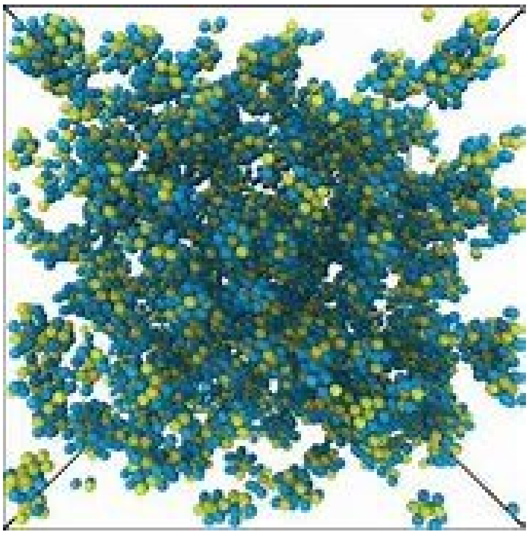


Figure 2.23: A percolation fractal

One can attribute very different physical properties to the points or the surfaces and thus model certain physical properties. Both types show that the connection between the elements can play a role. For example, one can consider edges between nodes or connected surfaces as electrically conductive. As long as there is no connection between opposite outer sides, the system acts as an insulator. However, if the conducting islands connect two opposite outer edges, it is an insulator.

Ladder. For this, there must be at least five edges from left to right or five black squares from left to right. But that is the minimum. It is easy to see that there are many structures where five elements are not enough. But there is also a threshold value that ensures that it is a conductive connection in every case.

Another example is the flow of water through a solid substrate. Minerals can be dissolved out; the particles that are washed out are called percolates. Percolation is also called leaching. This term is most commonly used in hydrology, where it refers to water percolating through the soil. Percolation only occurs when the maximum water storage capacity of the soil is exceeded in relation to gravity. If this happens over the entire thickness of the unsaturated zone, new groundwater is formed.

In pharmaceutical technology, percolation is used to extract active ingredients from plants. In this process, a mostly warm solvent, for example water or alcohol, is passed through the plants or plant parts. A well-known example of percolation is the preparation of filter coffee. In the production of spirits, percolation means the extraction of drugs by displacement. For this purpose, cylindrical vessels made of copper or stoneware are used, which taper downwards towards the outlet. The crushed drugs are placed between two sieves in the percolator and the 30 - 60 - percent preparation spirit is poured on top. The spirit diffuses and accumulates with the alcohol-soluble substances of the drugs within a few days. After three to six days, the prepercolate is slowly drained off. At the same time, the preparation spirit continues to be added via the inflow until the substances contained in the plant parts have essentially been extracted. This is followed by post-percolation, in which an attempt is made to squeeze out the remaining spirit by adding water.

In many countries of the world there is a rainy season and a dry season. Many bodies of water dry up completely. When the rainy season begins, smaller, separate puddles first appear. These join together when it continues to rain; if the rain continues, the water begins to flow, even to the point of powerful floods.

Currently, the methods for modelling epidemics are up to date. One can imagine individuals susceptible to the pathogen as nodes and contact between the individuals as edges. Above a certain population density, if there is sufficient contact between the individuals, the percolation threshold would be exceeded, i.e. large, contiguous clusters would form, leading to a spread of the pathogen to larger areas of the population. Empirically, the existence of such a percolation threshold was *demonstrated* using the *gerbil*, whose colonies have different population densities.

In daily life, many percolation-like phase transitions occur, e.g. the "pudding problem" (gel formation), the "cream stiffener problem" and the problem of the "clumping". In all cases, the effect only approaches the desired or undesired maximum when a critical value of the causal parameter is exceeded, usually according to a power law with a critical exponent, whereby the maximum effect initially increases very rapidly when the critical value is exceeded. Chemical additives, such as pudding or "cream stiffener" powder, can be used to reduce the critical value without, however, changing the principle.

Figure 2.24: Forest fires in Siberia

When analysing satellite photos of three devastating forest fires in Europe, Gui- do Caldarelli from La Sapienza University in Rome noticed their similarity to fractals. Indeed, many forest fires seem to have the typical fringe pattern of so-called percolation clusters at their edges. Another indication of the fractal growth of the fires is the fact that usually not the entire area enclosed by the fire front burns, but rather non-burning islands remain. Caldarelli took these two characteristics typical of percolations as an opportunity to analyse forest fires in simulations for fractal growth. The images he generated in this way show a high degree of agreement with the satellite images. Therefore, the spread of forest fires indeed seems to obey fractal growth. Extinguishing actions should therefore concentrate on the peripheral areas of a fire - there the fractal growth, i.e. the spread of the fire, can be stopped more effectively than by attempting to extinguish the hearth. Fig. 2.24 shows that no limit exists for fractals. An interstellar gas cloud is shown, which can lead to the formation of new stars.

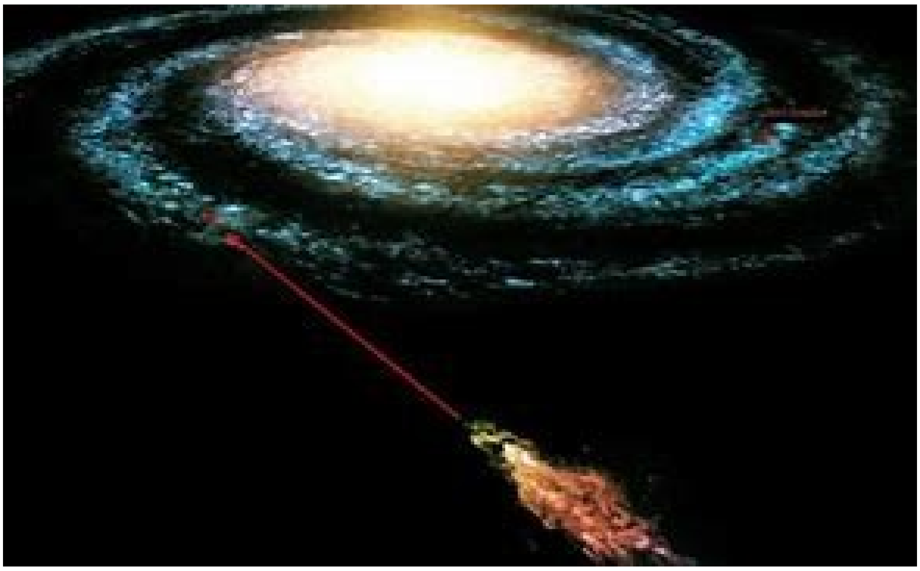


Figure 2.25: An interstellar gas cloud

Another form of percolation is invasion percolation. It provides a model for the mutual displacement of fluids that can be observed, for example, in the production of crude oil.

Figure 2.26: Pollution of an environment by spilled oil
(https://stopoilcontamination.blogspot.com/p/risiken.html)

The Jerusalem Cube is a solid fractal discovered by Eric Baird. Its construction is similar to that of the Menger sponge. Each iteration produces two self-similar elements of rank $n+1$ and $n+2$. It is so named because of its similarity to the Jerusalem Cross.

Its Hausdorff dimension is $D \approx 2,529$. Again, one sees the interesting super-

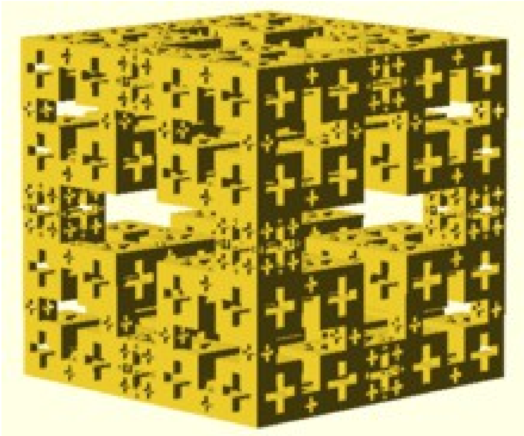


Figure 2.27: The Jerusalem Cube

behaviour. The dimension expresses that this is more than a surface, but that it is not quite enough to be a body. You can now try this out with m a n y other bodies; they are arranged according to increasing Hausdorff dimensions.

- Fractal icosahedron: $d = \frac{log(12)}{log(1+\phi)} \approx 2, 5819.$
- Fractal octahedron: $d = log_2 6 \approx 2, 5849.$
- Cooking surface: Each equilateral triangle is replaced by 6 smaller triangles, $d = log_2 6 \approx 2, 5849.$
- Greek cross: $d = log_2 6 \approx 2, 5849.$

Extensions of the Hilbert, Lebesgue, Moore and Mandelbrot curves have the Hausdorff dimension 3.

In kite curves you can see very well how a flat surface is slowly filled in by short straight-line movements. The following drawing shows which sequence of steps leads to a kite curve.

The curve is created according to this drawing by a sequence of two steps each. turning to the right or left after one step. There are different ways of combining the number of steps, for example once to the right, twice to the left, three times to the right, etc. If, for example, you constantly turn the sequence "up, then right", "down-then left", the curve quickly disappears into infinity, you can also easily find sequences in which only squares or rectangles are created, and so on. Tile patterns can be created quite easily.

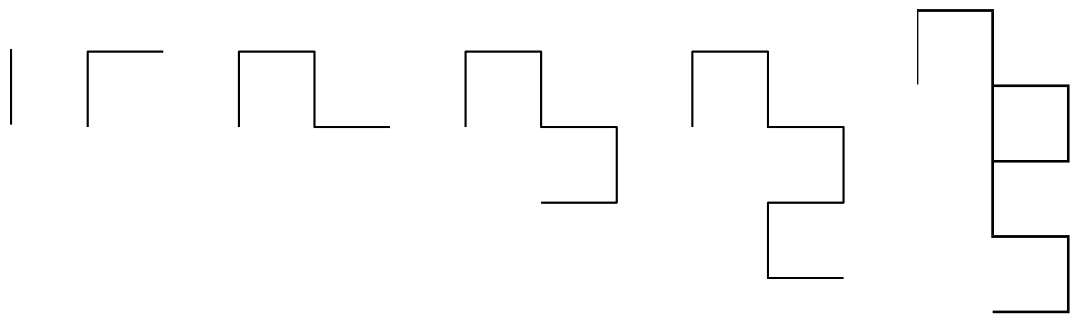


Figure 2.28: Kite curve

2.9 The growth of fractals

The growth of aggregates is one of the most interesting points to study in fractals. One distinguishes the following processes:

- the aggregation,
- the connection of particles (clustering),
- the agglomeration,
- the formation of tightly bound aggregates (which do not disintegrate again).

In everyday life, one finds many examples of aggregate growth:

- the electrode position,
- the formation of aerosols (e.g. soot, pigments),
- Colloids,
- biological growth processes (in plants and animals),
- the growth of rock inclusions,
- Displacement of liquids,
- delectrical carbon copies.

The simplest growth model is the Eden model. Mould on a damp wall is an example of this. Mosses and lichens develop according to the same

Principle.

It works according to the following algorithm.

- The first particle of the aggregate is placed at the coordinate origin of a given square or cubic grid and represents the seed of the aggregate.
- In the following steps, additional particles are placed in randomly selected locations adjacent to the aggregate formed in the previous step.
- Eden clusters have an irregular surface.
- Initially existing cavities inside the cluster are filled over time; the structure is compact inside:
- There is no scale variance; the structure is only self-affine.

Diffusion-limited aggregation (DLA) (Whitten and Sander, 1981) can be used to describe electrodeposition, fluid displacement, electrical breakthroughs (Bltze) and bacterial colony growth.

The grid version of the DLA model works according to the following algorithm.

- The first particle of the aggregate is placed at the coordinate origin of a given square or cubic grid and represents the seed of the aggregate.
- Further particles are released at random locations that lie on a circle or on a spherical surface. r_l is the radius of the sphere.
- The released particles then perform stochastic jumps on the lattice sites and diffuse through the lattice.
- If these particles reach grid sites adjacent to the aggregate during diffusion, they will stick to the aggregate
- The attachment is irreversible, it is a strong bond.

- New particles are released until the aggregate has reached the desired size.

DLA clusters are self-similar. The fractal dimension of the DLA cluster is equal to 1.66.

There are a number of variants of this DLA model.

1.) The DLA model is combined with reaction-controlled attachment. Particles adhere to the aggregate with an attachment probability W < 100 %. One can often operate with very low probabilities: W = 10 %, W = 5 %, W = 1 % give very different pictures. DLA clusters grown at different attachment probabilities W become more compact as the attachment probability W decreases. Particles can diffuse deeper into the centre of the cluster before they are finally trapped. Fractal dimension is greater than 2 for W > 0.

2.) Other boundary conditions For example, one considers particles diffusing from one side of a plate to aggregates growing on the opposite, absorbing side of the plate.

Agglomeration of gold colloids is used when gold particles are in a suitable solution. The agglomeration of the gold particles occurs through van der Waals interactions. The van der Waals forces are weak interactions that occur between different atoms or molecules. They are intermolecular forces. The van der Waals interaction is not a real bond. It exists between almost all atoms and molecules, but is mainly covered by stronger bonds.

Random paths occur when there is no control at all. For a process in the plane, one proceeds in the following way:

- the particle is set free at the coordinate origin;
- the particle wanders aimlessly in random directions on the grid to the next grid point.
- Once at the grid point, the particle again travels in an arbitrarily chosen direction to the next grid point.
- The last step is repeated as often as desired

We will go into more detail in section 2.10. It follows the rules presented here. Other examples are the pollen grains in the air, which cause discomfort for many people in spring, smoke particles or dust in the air, ink in the water and many more.

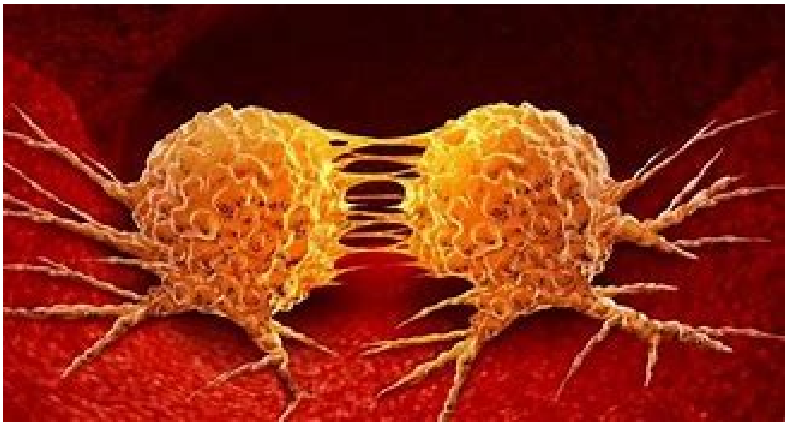


Figure 2.29: The growth of cancer cells (https://www.heilpraxisnet.de/naturheilpraxis/krebs-neuer-wirkstoff-stops-tumour-growth-20200627517292/)

Very typical growth processes caused by liquids and easily soluble minerals are `stalactites` and `stalagmites` in underground caves. A stalactite is the stalagmite that hangs from the ceiling of a cave, its counterpart is the `stalagmite that` grows up from the floor.

Like all dripstone formations, stalactite is formed when carbonated water penetrates the rock and, due to surface tension, deposits calcite on the ceiling of a cavity. The resulting material is called `sinter.` If a drop always emerges in exactly the same place due to the surface shape, the deposit can form the shape of a ring with the drop size as its diameter. This formation can grow into a sinter tube.

A stalactite is formed by drops no longer running down the inside of the tube but on its outside and depositing calcite; this causes growth in width. When examining a stalactite, the sinter tube can often still be detected inside. There are also rarer forms in which a stalactite grows without a previous sinter tube; in this case, the surface shape of the cave ceiling plays the decisive role so that the thickness growth begins without a guiding structure. The size of a stalactite is limited by its own weight, which eventually tears it off the ceiling. A stalactite is always significantly thinner than a stalagmite at the bottom. If the stalactite finally grows together with the stalagmite, a stalactite column forms, called stalagnate. The largest stalactite in Europe discovered to date, measuring more than seven metres, can be found in Doolin Cave near Doolin in County

Figure 2.30: The world's largest stalactite

Clare, Ireland. It was discovered in the 1950s by two adventurous people and opened to the public in 2006. Stalactites and other stalagmites can also form on older structures when calcium hydroxide is dissolved from cement or concrete and then reacts with carbon dioxide in the air.

A stalagmite is a stalactite that grows from the floor of a cave, its counterpart is a stalactite that hangs from the ceiling. One speaks of stalagmite when both types have grown together. A stalagmite is a speleothem in which dripping carbonated water, which deposits calcite, creates a stalactite that can take on different forms. The intensity of the dripping, but also the height of fall of the dripping water and the nature of the ground have an influence on the shape of the stalagmite. Thus, a distinction is made between the uniformly slender candle stalagmites and the cone-shaped palm trunk stalagmites. The candle stalagmites are formed by an even supply of solution and can reach a height of several metres with a small diameter. The cone-shaped stalagmites are formed by a very strong seepage water supply and can be several metres in diameter at the base, such as the Millionaire in the Sophien Cave.

The drop height in turn has an influence on the upper end of the stalactite. A rounded end is created when the water has a low fall height; as the fall height increases, the end becomes flatter and, in extreme cases, can be curved inwards. Furthermore, there are stalactite forms that deviate from these basic values, such as a stalagmite in the Schulerloch, the top of which is shaped as a water basin, or the Vesuvius in the Eberstadt Stalactite Cave, which was formed by carbonated water but is now slowly being eroded again by non-carbonated water.

Figure 2.31: The fairy caves near Saalfeld

2.10 Brownian Movements

Diffusion-limited aggregation (DLA) is the process by which particles undergoing random motion due to Brownian motion combine to form aggregates. This theory, proposed by T. A. Witten Jr. and L. M. Sander in 1981, applies to aggregation in any system in which diffusi- on is the main means of transport. DLA can be observed in many systems, e. g.
B. in electro-deposition, Hele-Shaw flow (the flow of a liquid between two plates with a very small distance), mineral deposits and dielectric breakdown.

The Brownian movement is the movement initiated by the Scottish botanist Robert Brown (1773
- 1858) discovered in 1827 under the microscope irregular and jerky thermal movement of small particles in liquids and gases. According to the explanation given by Albert Einstein in 1905 and Marian Smoluchowski in 1906, the displacement of the particles visible in the microscope is caused by the invisible molecules in the environment carrying out a thermal movement and, as a result of this disordered thermal movement, constantly bumping against the observed particles in large numbers and from all directions, with the result that, purely by chance, sometimes one direction, sometimes the other direction comes to bear more strongly. This idea was quantitatively confirmed in the following years by the experiments and measurements of Jean Baptiste Perrin (1870 - 1942). The successful explanation of Brownian motion is considered a milestone on the way to the scientific proof of the existence of molecules and thus of atoms.

The clusters formed in DLA processes are called Brownian trees. In 2D, these fractals have a dimension of about 1.71 for free particles that are

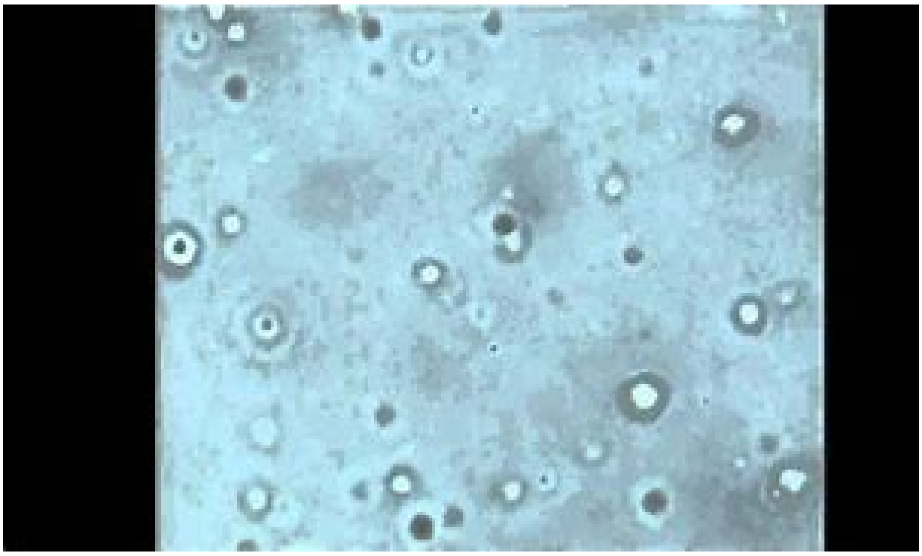


Figure 2.32: Milk on cold water

are not constrained by a lattice. However, in computer simulations of DLA on a grid, the fractal dimension changes slightly for a DLA in the same embedding dimension. Some variations are also observed as a function of the geometry of the growth, whether from a single point radially outwards or, for example, from a plane or line.

Computer simulation of DLA is an important means of investigating this model. Several methods are available for this purpose. Simulations can be carried out on a grid of arbitrary geometry and embedding dimension (this has been done in up to 8 dimensions), or the simulation can be carried out more along the lines of a standard molecular dynamics simulation, where a particle is free to move until it reaches a certain critical region, whereupon it is pulled onto the cluster. It is crucial that the number of particles in Brownian motion in the system is kept very low, so that only the diffusive nature of the system comes into play.

A Brownian tree is built in the following steps: First, a seed is placed somewhere on the screen. Then a particle is placed at a random position on the screen and moved randomly until it bumps against the seed. The particle is left there, and another particle is placed at a random position and moved until it bumps against the seed or a previous particle, and so on. The resulting tree can have many different shapes, depending essentially on three factors:

- from the position of the seedling;

- from the initial position of the particle (somewhere on the screen, in a circle around the germ, at the top of the screen, . . .);
- from the movement algorithm;

The colour of the particles can change between iterations, which leads to interesting effects. You can find a number of programmes on the internet that simulate this procedure, whereby you can set a whole range of parameters yourself.

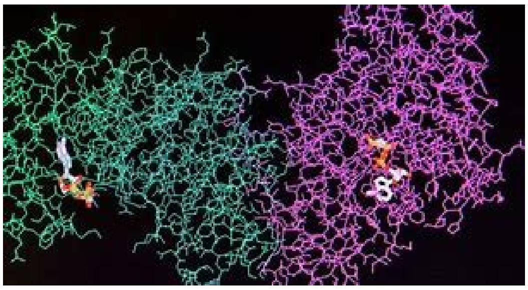


Figure 2.33: Clumping inside a body (https://www.esanum.de/today/posts/protein-steuert-verklumpung- der-blutplättchen-bei-thrombose-und-schlaganfall)

In medicine, the constriction or occlusion of vessels and organs plays a major role. Here one can observe very clearly how the accumulation of individual negative elements (for example, carbon particles inhaled during smoking) can lead to an increasing constriction of important vessels, which are ultimately always life-threatening.

Underground sewage systems are particularly susceptible to such blockages if they are not constantly cleaned and inspected (Fig. 2.34).

Figure 2.34: The grease mountain in a London sewer

One of the largest fat mountains of all time was found underground in London in 2017. It weighed 130 tonnes (Fig. 2.34).

Diffusion-limited growth was described by Leonard M. Sander and Thomas Witten in 1981. They had become aware of the observation of such structures reminiscent of Mandelbrot's fractals in deposits of iron colloids and tried to simulate their formation in a simple computer model. In the Witten-Sander model, a particle moves predominantly under the influence of diffusion (Brownian motion) until it comes close to a "cluster" of already deposited particles. If it falls below a minimum distance, it attaches itself to the cluster. To the surprise of Sander and Witten, the simulation resulted in branched, fractal structures.

The accumulation of soot particles, known from all areas where lignite-fired power plants operated, is similar: Particles accumulate on the walls of a chimney, on house walls, window panes and shirt collars. Other examples are

- Precipitates in electrolytic solutions, e.g. copper sulphate solution, which are precipitated to pure copper by suitable cathodes,

- Tree-like structures in biology, such as the formation of fur markings in zebras, tigers, leopards or tapirs.

- Lichtenberg figures (Fig. 2.34) are typically formed by the electrostatic discharge or redistribution of electrical charges on the surface of insulator plates. The physical principles underlying the formation of Lichtenberg figures are the same as those on which modern electrophotography (xerography) is based, which is used in all copying machines in common use today (photocopiers, laser printers, etc.) (Fig. 2.35).

Figure 2.35: A Lichtenberg figure

Figure 2.36: Charges between two conductors

2.11 Cracks and Fractures

The fracture of solid bodies is usually produced by an external load (Fig. 2.37). Often ageing processes or advanced corrosion are to blame (Fig. 2.38). However, a load exceeding a permissible or tolerable limit can also be responsible. Humans are relatively helpless against the forces that occur during earthquakes. For high-rise buildings in areas at risk from earthquakes, many technological improvements have already been found that reduce or avoid possible damage. Broken glass panes are almost part of everyday life. Often, collapse hazards also occur when underground mining has been going on for a long time in a certain area. Fracking is still hotly contested.

In fracking, a fluid (frac fluid) is injected through a well under high pressure, typically several hundred bar, into the geological horizon from which the well is to be extracted. The frac fluid is water, which is usually mixed with proppants, such as quartz sand, and thickeners. Usually, several deflected boreholes (laterals) are first drilled in the target horizon using directional drilling, whereby the drill bit is guided parallel to the layers. As a result, the available borehole length in the reservoir is considerably greater, which generally increases the production yield. High-volume hydrofracking uses large quantities of fluid with more than 1000 m3 per fracking phase or a total of more than $10{,}000 m^3$ per borehole.

Since the end of the 1940s, fracking has been used primarily for oil and gas production, as well as for tapping deep aquifers for water extraction and improving heat transport in deep geothermal energy. In the latter applications, no proppants or chemical additives are needed. Since the beginning of the 1990s, and especially in the USA from around the year 2000, extraction by means of fracking has focused on so-called unconventional crude oil and natural gas. The fracking boom there changed the US energy market considerably and caused prices to fall. This led to a debate about the profitability of the process. The US government has therefore been supporting

2013 Efforts to increase the export of liquefied natural gas to Europe and Japan, including accelerated licensing procedures.

While some voices emphasise this geostrategic component through the change in international dependencies, the environmental risks and possible health hazards of the "fracking boom" lead to a controversial and still ongoing technical, political and social debate, especially in Europe. Some countries and regions have legally banned natural gas fracking on their territory.

Figure 2.37: A broken ice sheet

Cracks and fractures can be studied at different levels. At the level of electrons or the displacement of grain positions ($^{10-6}$ cm), materials science or materials$\leq$science are responsible. The terms materials science and materials science are closely linked: Materials science, with a more natural science approach, deals with the production of materials and their characterisation of structure and properties, while materials engineering includes engineering-oriented materials development as well as the corresponding processing procedures and the operational behaviour of components in use. Both sub-areas encompass research activities of the most diverse material classes and material development chains.

An essential feature of materials science and materials engineering is the consideration of the structural composition of materials and the mechanical, physical and chemical properties that depend on it. This includes the characterisation, development, production and processing of construction materials and functional materials.

The department is made up of knowledge-oriented basic research on materials and engineering materials development with a focus on applications. In doing so, it develops a strong leverage effect in the sense of a

Implementation of research results in market-relevant innovations. At the same time, as an interdisciplinary science, materials science and materials engineering have a far-reaching integrative effect in that they take up findings from neighbouring disciplines and have a reciprocal relationship with them. For materials science, the links with chemistry, physics and the life sciences should be mentioned in particular, while for materials engineering the fields of mechanics, design engineering, production engineering and process engineering are relevant.

If the orders of magnitude considered are $> 10^{-1}$ cm, then the prevention of fractures and similar damage is a subject of engineering science. In the orders of magnitude in between, `applied mechanics` plays a major role (Fig. 2.37).

Fractures, cracks, splintering, collisions, etc. mostly show fractal structures.

Figure 2.38: A crack in an old wall

Figure 2.39: A crack in a car tyre

Cracks in the road are continuously washed out by incoming surface water. The road loses its substance, and the additional alternation between frost and rain and thawed soil leads to a loosening of the entire structure. Therefore, open cracks inevitably lead to greater damage (Fig. 2.40).

Figure 2.40: Cracks in a road

The geological processes on Earth provide many fractal images: the collision of continental plates, the uplifted mountains, the volcanoes along the edge of the collision all have a fractal structure (Fig. 2.41 and Fig. 2.42).

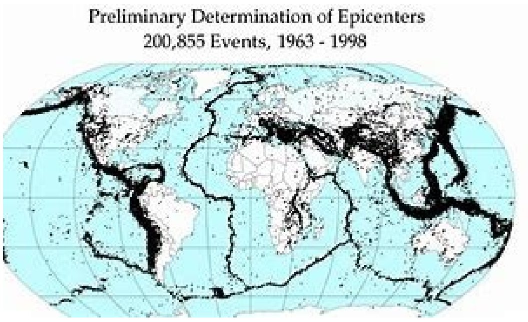


Figure 2.41: The collision of continental plates

Figure 2.42: Flowing or solidifying lava has a fractal structure

3 Fractals in Medicine

The brains of small mammals are relatively smooth, whereas those of humans are highly wrinkled. A fractal dimension between 2.79 and 2.73 seems to be typical for the human brain. Fractal structures are also found in the membranes of liver cells. The nasal bones of deer and arctic foxes ensure maximum olfactory sensitivity by packing the largest possible surface area into a small volume. This results in a fractal structure with constant broken dimensions. The human blood supply branches at least eight times, the maximum number is about 30, thus reaching all body cells; the fractal dimension is equal to three. Fractal self-similarity pervades the bodies of organisms, but it is not the flat homunculus-like self-similarity imagined by earlier science. The body is a network of self-similar systems such as the lungs, the vascular systems, the nervous systems.

3.1 Encephalography

Brain waves also follow fractal patterns. However, measurements during epileptic seizures have revealed unusual regularities here (Fig. 3.1). Depression is one of the main causes of health impairment. EEG measurements can provide one or more economic biomarkers for the dia- gnosis of depression. Here, we compare frontal alpha asymmetry (FAA), posterior alpha asymmetry (PAA) and Hi- guchi fractal dimension (HFD) in terms of their ability to predict PID-5 depression and the specificity of these predictions compared to PID-5 anxiety. University students provided 8- or 10-minute resting EEG scores and questionnaire scores on PID-5 depression and PID-5 anxiety. FAA and PAA did not show significant correlations with the measured values at any electrode pair. The HFD was based on different frontal and posterior factors, which correlated significantly with anxiety and with each other. Posterior HFD also correlated significantly with depressiveness, although this correlation was weaker than that with anxiety. The proportion of the variance in d e p r e s s i v e n e s s that was attributable to posterior

HFD was not unique, but was shared with anxiety. The inclusion of patients with anxiety disorders in the sample meant that the frontal factor was slightly more predictive than the posterior factor, but in general it confirmed the previous conclusions. In contrast to our predictions, none of our measures specifically predicted depressiveness. Previous reports of associations with depression may be related to concurrent anxiety. The HFD may be a better measure of anxiety than depression, and the earlier association with depression may be due to confounding between the two, as depression is very common in severe anxiety.

Figure 3.1: Recordings of an EEG device

Increased EEG synchrony leads to a reduction of the fractal dimension. This has been observed in epileptic seizures, which show increased synchronous oscillations and a decrease in arrhythmic activity, and under anaesthesia. In contrast, increased complexity is observed in manic phases of bipolar disorder and in the frontal lobes of schizophrenia patients. Both bipolar disorder and schizophrenia are characterised by disturbed syn- chronic activity. The Higuchi fractal dimension (HFD) is a method for calculating the fractal dimension of a time series and is used in the present analysis. A small number of studies have examined the HFD in depressed populations, all of which found a higher value, suggesting that the HFD may be useful as a biomarker for depression.

Epilepsy is one of the most common diseases of the brain. In the clinic, to confirm an epilepsy-like symptom and to localise the seizure, electroencephalogram (EEG) data is often visually examined by a clinician to detect the presence of epileptic discharges. Epileptic discharges are transient waveforms that last from tens to hundreds of milliseconds.

and mainly divided into seven types [**5**]. It is important to develop systematic approaches to accurately distinguish these waveforms from normal control waves. This is a difficult task because clinically used scalp EEGs usually contain a lot of noise and artefacts. To solve this problem, 640 multi-channel EEG segments, each 4 seconds long, were analysed. Of these segments, 540 are short epileptic discharges, and 100 are from healthy controls. Two approaches have been proposed to distinguish epileptic discharges from normal EEGs. The first method is based on the signal range and correlation properties of the EEGs. The second method is based on networks constructed from three aspects of the scalp EEG signals: the signal range, the energy of the alpha wave component and the long-term correlation properties of the EEG. The networks are further analysed using singular value decomposition (SVD).

The EEG data analysed in this study were obtained from the First Affiliated Hospital of Guangxi Medical University. The studies with human participants were reviewed and approved by the Ethics Committee of the First Affiliated Hospital of Guangxi Medical University. The participants gave their written consent to participate in this study. Fifty-nine epilepsy patients underwent three hours of video EEG monitoring with 19-channel EEG recording with scalp-mounted electrodes using the international 10-20 system at a sampling rate of 256 Hz. The electrode impedances were kept below 10 kilΩ. The 19 electroencephalographic scalp electrodes were arranged according to the designations Fp1, Fp2, F7, F3, Fz, F4, F8, T3, C3, Cz, C4, T4, T5, P3, Pz, P4, T6, O1 and O2.

All epileptic discharges were annotated by an experienced clinical neurophysiologist based on the average montage with an analogue bandwidth of 0.1 to 70 Hz and a notch filter of 50 Hz. The EEG signals were segmented into 4s epochs and randomised for each participant. The collected epochs were converted into the European Data Format (EDF) for further analysis. A total of 532 EEG recordings of epileptiform discharges were collected from all participants and 100 healthy controls, each lasting 4 seconds. Among the 532 brief epileptic discharges were 69 spikes, 82 sharps, 174 spike and slow wave complexes, 72 sharp and slow wave complexes, 64 polyspike complexes, 77 polyspike and slow wave complexes and two rhythmic spike discharges.

- Spike: Spikes are the most basic EEG activity with a duration of 20 to 70 ms. The amplitude varies and is typically >50 μV .

- `Sharp`: A sharp wave resembles a spike and has a duration of 70 to 200 ms (5 to 14 Hz). Its amplitude varies between 100 and 200 μV , and the phase is usually negative.

- `Spike and slow wave complex`: An epileptic pattern consisting of a spike and an associated slow wave following the spike that is distinct from the background activity; it may be single or multiple.

- `Complex of sharp and slow waves`: An epileptic pattern consisting of a sharp wave and an associated slow wave that follows the sharp wave and is distinct from the background ac- tivity; it may occur once or several times.

- `Polyspike complex`: A sequence of two or more spikes.

- `Polyspike and slow wave complex`: An epileptic pattern consisting of two or more spikes associated with one or more slow waves.

- `Spike rhythm`: Refers to a widespread burst of a spike rhythm at 10 to 25 Hz, with an amplitude of 100 to 200 *muV* and the highest voltage in the frontal area, lasting more than 1 s.

Often epileptic discharges are related to an amplitude greater than that of a normal control EEG. The signal range is calculated as follows:

$$max\, x\,(t),\ t \in [t\ ,\ t_{12}\ ]\ \text{-}\ min\, x\,(t),\ t \in [t\ ,\ t_{12}\ ], \tag{3.1}$$

This procedure is applied to each of the 19 EEG signals with respect to the earlobes (i. e. i.e. to the difference of the EEG signals measured at the 19 electrodes and the earlobes) or to the difference of the EEG signals according to the network design. In the first case, the final signal range is estimated as the average of the 10 largest signal ranges obtained from the 19 EEG signals.

In clinical applications, brainwaves are often divided into five bands: Delta (0.5 to 3 Hz), Theta (4 to 7 Hz), Alpha (8 to 13 Hz), Beta (14 to 30 Hz) and Gamma (> 30 Hz). The alpha wave is most clearly visible when the person is relaxed and has their eyes closed. It has been found that the alpha wave component is often larger in the back of the head during epileptic discharges. To calculate this component, a Fourier transform of the EEG signal is used; from this, the power spectral density (PSD) is calculated and finally the PSD curve is integrated over the alpha wave band.

Adaptive Fractal Analysis (AFA) uses an adaptive detrending algorithm to extract globally smooth trend signals from the data for a given time scale, and then analyses the scaling of the residuals of the fit as a func- tion of the time scale [**6**],[**7**]. The main steps of AFA to estimate H are as follows:

It is assumed that starting from a stationary incremental process $x(1), x(2), x(3), \ldots$ a random walk is constructed by the following equation:

$$u(n) = \sum_{k=1}^{n} \left(x(k) - \bar{x} \right), \quad n = 1, 2, 3, \ldots N. \tag{3.2}$$

where $\bar{x}$ is the mean value of the process.

Starting from this random walk $u(n)$, a global trend $v(i)$, $i = 1, 2, \ldots N$ for any time scale w can be determined, where N is the length of the original time series. This is achieved by dividing the above random walk process into overlapping windows, where the size of each window w contains an odd number of samples and adjacent windows overlap by $(w+1)/2$ samples. The random walk process in each window is fitted by a polynomial of order M, and the polynomials in overlapping regions are combined to obtain a single global trend. Typically, M should be equal to 1 or 2, a linear or quadratic function. Local fitting ensures that the global trend is optimal or close to optimal, as a local Taylor series expansion is used.

After obtaining the global trend $v(i)$ of $u(i)$ by the above method, the residual $u(i) - v(i)$ can describe the variation around the global trend. For fractal processes, the Hurst exponent H can be calculated by the following equation:

$$F(w) = \left[\frac{1}{N} \sum_{i=1}^{N} \left(u(i) - v(i) \right)^2 \right]^{1/2} \approx w^H. \tag{3.3}$$

The above equation means that by calculating the variance of the residual between the original random walk process and the fitted global trend under a varying window w, one can obtain a linear (or multiple linear) relati-
between $\log_2 F(w)$ and $\log_2 w$.

A whole series of algorithms is applied to this initial situation, as a result of which a whole series of parameters is obtained that allow a more precise assessment of the situation. For these considerations, please refer to the article [**5**].

3.2 The heart

Leonardo da Vinci already observed that the walls of the human ventricles are lined on the inside with a complex meshwork of heart muscle fibres. Unlike the large blood vessels, the wall of the ventricles is not smooth, but rather spongy. The trabeculae form a multiply branched transitional layer between the massive heart muscle and the interior of the ventricles. In the early embryo, they play an important role in supplying the heart muscle. Before the coronary vessels are fully formed and supplied with blood, the large surface area of the trabeculae enables the diffusion of nutrients and oxygen from the blood into the heart muscle.

The muscle fibre networks on the walls of the heart chambers are fractally structured, they influence the heart's performance. The more complex the branches of the trabeculae are, the better the heart pumps. A higher fractal dimension is linked to a larger heart beat volume, stronger pumping power and better heart condition.

The human heartbeat follows a fractal rhythm. If the heartbeat and breathing rhythm become too regular, it can lead to heart failure due to congestion. However, if the rhythm becomes too irregular, it causes the fibrillation of a heart attack. In practice, the rhythm fluctuates back and forth between chaos and order throughout life.

Sixteen gene loci were identified that are closely linked to the cardiac muscle fibre structures and their shape. Ten of these gene loci additionally control other aspects of cardiac function, such as pulse rate, left ventricular structure and the duration of a rash complex in the ECG [**6**].

3.3 The eye

`Retinal implants` with fractal geometry could be suitable for helping patients who lose their vision due to retinal diseases. This is shown

Figure 3.2: The fractal structure of the human heart

Researchers led by physics professor Richard Taylor from the University of Oregon. According to the researchers, fractal implants stimulate healthy neurons in the retina much better than those with normal, flat electrodes. According to the researchers, this should make it possible for the first time to restore vision to such an extent that those affected can find their way around rooms and even on the street.

Figure 3.3: Retinal vessels - fractal structure

Retinal diseases such as macular degeneration threaten the eyesight of many people. The use of implants that still stimulate healthy nerve cells could bring relief. The team led by physics professor Richard Taylor is aiming for a new approach: changing the geometry of the implants from simple Euclidean shapes such as squares to fractal structures, which ultimately also correspond better to those of natural neurons.

Computer simulations now indicate that really drastic improvements can be achieved with this. Under ideal conditions in the simulation, it could be shown that a single fractal electrode stimulates all target neurons, while the Euclidean electrode only connects with ten percent. At the same time, the fractal electrode manages with less than half the voltage.

As promising as the simulation results are, there is still a long way to go before fractal implants could actually give blind people back their perception. According to Taylor, the data obtained in the simulations will help to first develop miniature versions of real implants. These are intended for tests on mice. So it may still be a long time before clinical tests on humans or even general approval.

3.4 Some conclusions

Diseases represent a chaotic state for the body. Through fractal conditions that bring this chaos back into a stable state, recovery can take place. This happens, for example, through the application of fractal `vibration patterns`. In this way, a system that has become unstable can be brought back to the desired range of stability.

Which oscillation pattern is required for recovery in each case can be determined with the help of a special mathematical formula. In general, these mathematical formulas describe stable trajectories as attractors for mechanical body movements; these are trajectories that are located in the three-dimensional geometric space. In analogy, stable states or areas can be defined for biological systems, which do not apply to the geometric space, but for example to the dimensions `frequency`, `phase` and `time`.

If, for example, the intake of medication is examined according to this fractal formalism, both in terms of time and quantity, it is shown that over time the effect increases so that the dose can be reduced.

When treating with vibration patterns, it seems advisable to distinguish between frequency patterns of building up and breaking down effects depending on the desired therapeutic effect. Building up effects are e.g. the cohesion and coordination of body systems and communication systems in the body. Degrading effects are, for example, the elimination of toxins or microorganisms. The therapy results can be significantly improved by this.

If the treatment with vibration patterns is carried out by means of light and colour therapy, the following questions are essential for the success of the therapy:

- How fast should the coloured lamps flash?

- How long should the on and off times of the lamps be?
- In which rhythms should the different colours run one after the other?
- With what intensity should the colours shine?

The importance of the answers to these and similar questions was demonstrated almost 80 years ago by Alexander Gurwitsch, the early pioneer of biophotons. In an experiment, he separated two cultures of yeast cells by an opaque disc that had holes in it. When the cultures could see each other continuously (a hole was located exactly between the cultures), there was a bilateral increase in the percentage of dividing yeast cells after six to eight minutes. When the disc was rotated at 50 hertz, this increase was already present after 30 seconds. The time could be reduced to 12 seconds when the disc rotated at 100 to 800 hertz. This showed not only that there is a mutual influence by biophotons, but also that this becomes much more effective when the flow of biophotons is interrupted with a certain frequency (pulsing). Correctly pulsed light or pulsed colours have a greater effect on organisms than uniform irradiation.

Commercially available devices are already offered, for example with the name OptiSanPro. As a specialised application, the device is successfully used by therapists for basic and advanced treatment. In all detox measures, the elimination of metals, petrochemical substances, phthalates, herbicides, pesticides, insecticides, environmental toxins and vaccine contamination can be supported and accelerated. Success has also been seen with allergens. Fasciae, the long underestimated wrappings of connective tissue, can often be the source and cause of unexplained conditions and disorders in chronic diseases. Here, too, therapists report good and rapid successes.

For the treatment, different programmes were designed for specific conditions or areas, e.g. torso down, torso up, heart, wound treatment, skin, tissue, hormone/membrane and blockade. Special attention and care were given to the calculation of the fractal components such as the light intensity and rhythms (pulse duration and pause times) of the individual coloured illuminants and their combinations. The corresponding programmes are stored and can be started easily. A microprocessor controls the stored programmes, some of which consist of thousands of different changes to the lamps. The LEDs used for this purpose react in a timely manner to the most varied combinations of light pulses, which are triggered by corresponding pau- ces.

are interrupted.

The unit has eight different LEDs and another one that works exclusively in the infrared range. In addition to controlling the individual colours, e.g. turquoise or yellow, any number of different colour nuances that the eye can perceive are available through different combinations of the intensities of red, green and blue. This great variety of colours is synonymous with a great variety of photons, which are effectively used with different rhythms. All LEDs also have the ability to emit colours with increased intensity in order to specifically enhance certain physiological effects. The persons treated find the nature of these fractal impulses very pleasant and beneficial. (Dr. Siegfried Kiontke, physicist and author)

3.5 The lung

Disease risks of smokers can be assessed by the degree of branching of the bronchi. The extent of lung damage due to smoking can be detected in time by quantifying the loss of branching complexity in the entire bronchial system. This is made possible by fractal analysis of CT images, which uses the fact that the lung is constructed like a fractal according to the principle of self-similarity from ever-shrinking copies of its bronchial branches [**7**].

Figure 3.4: A healthy lung and its appearance after ten years of smoking

The signs of the onset of chronic smoker's bronchitis (COPD) are thickened walls and thus narrowed diameters of the airways as well as an increasing loss of bronchial branches in the periphery of the lungs, in the smallest airways. In some smokers who do not yet show any detectable narrowing of the airways in an examination with a computer tomograph (CT), it is possible to detect

The study also found that smokers can still detect an increased risk of disease and mortality by using fractal analysis of CT scans to determine the a i r w a y f r a c t a l d i m e n s i o n (AFD). "A major advantage of fractal analysis of CT scans is that the extent of lung damage from smoking can be identified earlier by quantifying the loss of branching complexity throughout the bronchial system. This makes it easier to assess the patient's risk of developing chronic obstructive pulmonary disease (COPD)," explains Dr. med. Thomas Voshaar, Chairman of the Board of the Association of Pneumological Clinics (VPK) and Head Physician of the Lung Centre at Bethanien Hospital in Moers, referring to recent study results from the USA. For some time now, researchers and clinicians have been looking for a method to detect changes in the bronchial system earlier, which always occur first in the periphery of COPD and then usually remain unnoticed for a long time. Only when symptoms such as shortness of breath occur during physical exertion can restrictions also be detected in a lung function test.

The basis of the fractal analysis is the structure of the lung, which corresponds to a fractal in its structure. It starts with the trachea, which is divided into two main branches, each of which supplies oxygen to one of the two lungs. If you move along a branch, you will see that the lobed bronchi at the ends of the main branches branch out further into the segmental bronchi and then into smaller and smaller branches. Thus, after about 20 - 25 levels, we reach the smallest branches of the bronchi, the so-called bronchioles, which have an internal diameter of less than 1 mm and then divide again into microscopically fine branches (bronchioli respiratorii and alveolar ducts). These finally lead into the actual, breathing lung tissue with a total of around 300 million alveoli (alveoli). The alveoli, which are miniature balloons 0.1-0.2 mm in diameter, give the lungs their spongy appearance and are densely packed in a cluster and attached to the finest tubular branches (alveolar ducts and bronchioli respiratorii). Their wafer-thin walls are criss-crossed by a network of tiny blood vessels (capillaries), which enable a rapid exchange of respiratory gases. Each alveolus is surrounded by about 1000 capillaries.

In the recent study of over 8000 smokers [**8**], researchers from the University of Alabama at Birmingham found a clear correlation between the fractal dimension of the airways and the following parameters of lung health:

- The lower the branching complexity, the higher the degree of airway narrowing, the frequency of airway disease and the loss of lung function in terms of one-second capacity FEV1 (which indicates the maximum volume of air that the patient can exhale within one second).

Figure 3.5: The fractal structure of the blood vessels in the lungs

- Less clearly, but also tending to be pronounced, AFD is also related to patients' quality of life and physical performance (6-minute walking distance), as well as - in an inverse relationship - to the frequency of exacerbations.

"In the future, it would be very informative to also determine the fractal dimension of the airways during CT examinations of smokers, as a loss of complexity of the bronchial branching can provide a good indication of how severely damaged and thus susceptible to disease the patient's lungs actually already are".

3.6 Cancer diagnosis

Researchers led by Joachim Spatz, Director at the Max Planck Institute for Intelligent Systems in Stuttgart and Professor at the University of Heidelberg, have discovered that tumour cells and healthy cells differ from each other based on their fractal geometry. The scientists enlarged the edges of pancreatic cells and analysed their irregularities. By recording these mathematically, they determined the fractal dimension of the cell edge, which is a measure of the statistical distribution of the irregularities. Cancer cells have a higher degree of fractalisation than healthy cells because very irregular bulges of different sizes form on the cell surface during uncontrolled tumour growth. The researchers not only recognised from the fractal dimension whether a tumour cell was present, but also determined with 97 percent certainty which of two differently malignant pancreatic tumours was involved. "This

In this way, cancer cells can be distinguished much more precisely and quickly than with t h e method commonly used up to now".

Identifying cancer cells has so far been uncertain and time-consuming. Until now, cancer cells and their site of origin in the body have been identified by staining a cell sample with certain antibodies and biomarkers. However, the staining method has disadvantages: it requires numerous individual steps with costly antibodies and is therefore time-consuming and expensive. In addition, the currently used dyes cannot always make the finest differences between cells visible. Therefore, cancer can only be correctly diagnosed in 85 per cent of samples using this method. With the help of fractal geometry, Joachim Spatz's team not only detects cancer cells more reliably, but also much faster. This is because the cells can be examined under a microscope without having to be specially prepared. In order to be able to capture the details of the cell edges, Joachim Spatz's team uses a reflection contrast microscope. Instead of illuminating the sample from below like a conventional light microscope, the microscope used by the Stuttgart researchers measures the reflection of the light beam on the cell surface. This differs depending on whether the light hits a cell directly or first hits aqueous cell culture medium and then a cell. Based on the reflected light, even tiny structures at the cell edge can be examined.

"Analysing the fractal geometry of cell surfaces holds great potential for clinical diagnostics. Now the scientists are investigating how their method can be applied in practice. To this end, they are investigating different malignant cell lines and primary cells, i.e. cells that are obtained from human organs and, in contrast to tumour cell lines, can only be cultivated for a certain period of time. "The next step for us will be concrete collaborations with clinics to test the method directly on relevant tissue samples." [**17**]

3.7 The skin

Electronic components in the brain can already help people with severe diseases of the nerves and brain - such as Parkinson's patients: Here, the current pulses alleviate their typical paralyses and twitches. Brain surgeons also insert such electrodes into the brains of people with epilepsy or depression.

US researchers have built a three-dimensional model of the human genome for the first time. With it, they found out how the cell manages to build its two-metre-long hereditary sub-

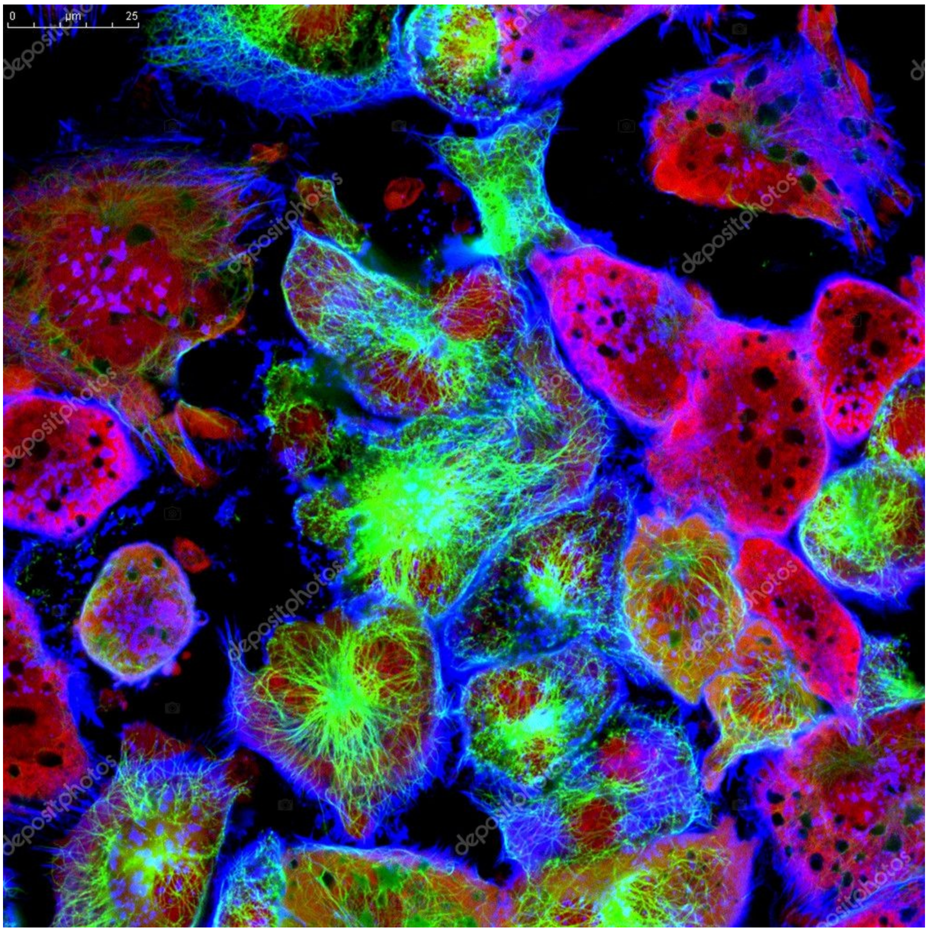


Figure 3.6: A far advanced tumour

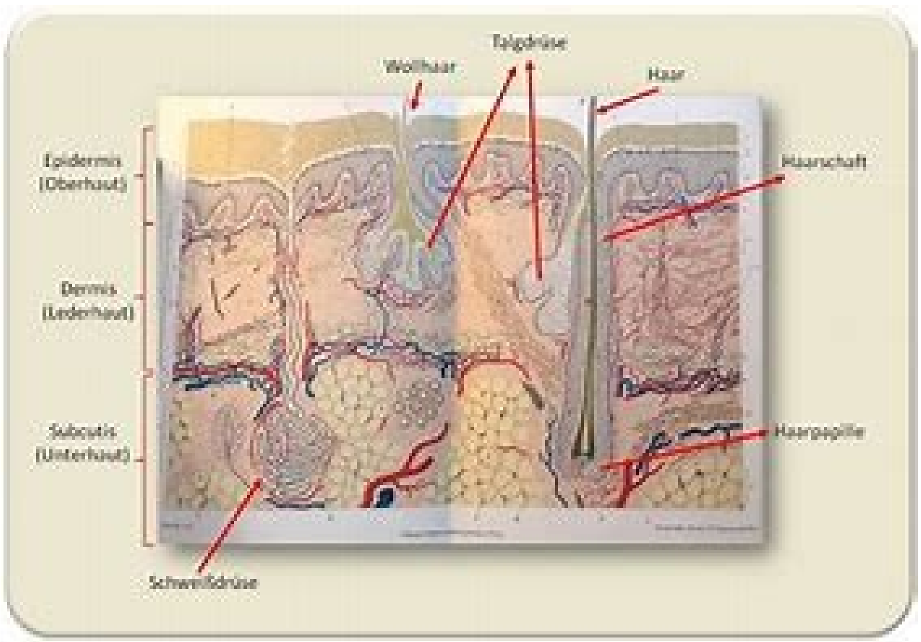


Figure 3.7: The skin also has a fractal structure

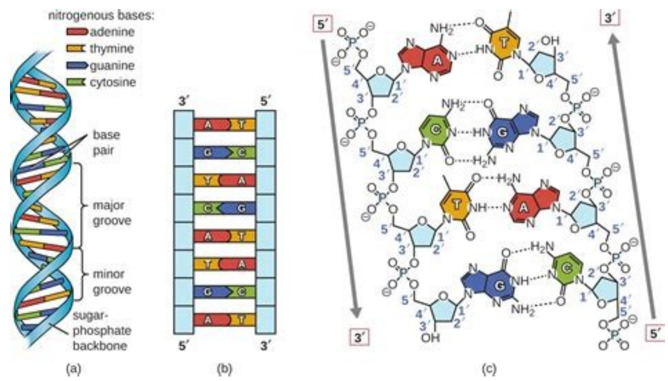


Figure 3.8: The fractal structure of DNA

stance in the cell nucleus. On the one hand, the cell organises two compartments, a kind of resting room and a factory, into which it sorts the active and the inactive genes respectively. On the other hand, the genetic material DNA forms itself into a pearl necklace-like structure, which in turn twists itself like a ball of wool. The information density of this spherical structure is much higher than that of a computer chip. At the same time, the organisation prevents the genetic material from becoming tangled and the cell from being able to read its own genome.

Until now, it was not clear how the strands of genetic material with their more than three billion DNA base pairs organise themselves in such a way that they fit into a human cell nucleus with a diameter of one hundredth of a millimetre. To make the spatial structure of the genome visible, the scientists now glued strands of DNA that were close together in the cell nucleus and then identified them. The genome is thus broken down into millions of pieces and then reassembled into a spatial map that shows neighbourly relationships. The 3-D model shows that the cells distribute their genetic material in two compartments, reports Job Dekker, a systems biologist at Harvard Medical School. In the first are the active genes, easily accessible to proteins and other control elements. The second compartment contains the passive DNA, which is packed very tightly together for this purpose. The individual DNA molecules switch between the factory and the resting room as required, with the respective active regions moving closer together.

For the storage of information, nature has created an additional extraordinarily dense node-free structure: DNA clusters together in the so-called fractal sphere without being hindered in unfolding for cell division. The fractal sphere architecture had already been discussed as a theoretical possibility more than 20 years ago. Only the development of the new method, which reveals the neighbourly relationships of individual genes, has now provided clarity.**[18]**

DNA is the carrier of the genetic code, and from it are formed all the genes that are responsible for the structure of our body. For a long time, it was believed that this was done exclusively by biochemical means. The DNA forms a huge double strand of bases in which the genetic information is encoded, with the help of which proteins can then be produced inside the cell.

Russian scientists have discovered, however, that DNA can do much more. Nearly 90 percent of this molecule is not needed at all for protein synthesis, but serves for communication and as information storage. The characteristic shape of the double helix makes DNA an ideal electromagnetic antenna. On the one hand, it is elongated and thus a rod antenna that can absorb electrical impulses very well. On the other hand, seen from above, it is ring-shaped and thus a very good magnetic antenna. In this way, our DNA can absorb electromagnetic radiation (light) from the environment. The absorbed energy is simply stored in the DNA by causing the molecule to vibrate, with a natural frequency of 150 megahertz.

According to the research results of Pyotr Garyaev and his team, the DNA is not only a transmitter and receiver of electromagnetic energy, but also absorbs the information contained in the radiation and interprets it further. DNA is therefore a highly complex interactive light-based biochip with 3 gigabits of storage capacity, which is also capable of understanding human language.

3.8 Fractals in the Anatomy

Many organs and structures of the human body, for example the retinal vessels, the respiratory tract, the renal arteries, the blood vessels supplying the heart, the liver and the dendrites of nerve cells can be viewed as fractal objects. Fractal geometry is used in cardiology to calculate heart rate, in neurology to analyse changing patterns of electroencephalograms (EEG) and in radiology to analyse bone healing, breast lesions and tomographs. Fractal geometry could also be applied in histopathology and cytology. There it is used to calculate several neoplasms such as gallbladder, lung, uterus, breast, oral cavity or larynx cancer. Recently, fractal methods have also been used to describe the complexity of surface topography. Fractal geometry has proven to be a suitable tool for measuring the irregularity and complexity of tissues at several levels: from the organisation of cells at the organ level to the surface of cells, the distribution of protein aggregates in the membrane, the organisation of the cytoplasm, the nucleus and the structure of individual proteins.

Anatomy education is currently being converted to 3D printer-generated models in many places. The need for real dead bodies is decreasing (Fig. 3.9).

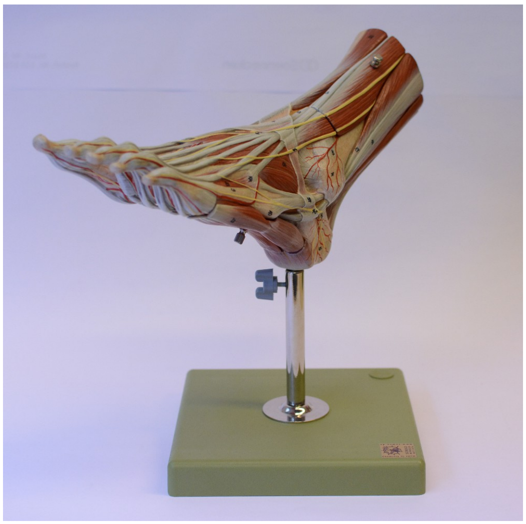


Figure 3.9: Model of a foot (for anatomy training)

3.9 Epidemics

An `epidemic` is a temporally and locally limited increased occurrence of cases of disease of uniform cause within a human population and thus corresponds to a large outbreak of a disease. The term is not limited to infectious diseases. The spread patterns of an epidemic also have a fractal appearance. They strongly resemble the spread of forest fires. They start at one or more points and spread in small steps to all sides.

In `epidemiology`, we speak of an epidemic when the number of new cases of the disease (`incidences`) increases over a certain period of time in a certain region. A distinction is made between `explosive` and `tardive epidemics` according to the speed of the increase in the number of cases. If the disease spreads to other countries and continents, it is referred to as a `pandemic.` A decline in the incidence of disease is referred to as a `regression.` An `endemic disease,` on the other hand, is `defined` as the persistently frequent occurrence of a disease in a circumscribed population; in this case, the incidence remains approximately the same.

is the same, but is higher than in non-endemic areas. In contrast to an endemic disease - in which a disease occurs continuously within a population with approximately the same number of cases (reproduction rate = 1) - an epidemic spreads with a reproduction rate greater than 1. In the case of an infectious disease, this means that the number of infected persons increases and the number of new infections rises. The rate at which new infected persons occur through contact with infected persons is significant for the spread; at the beginning, it corresponds to the basic reproduction number R0. Initially, the number of new infected cases per time interval increases by an approximately equal proportion compared to the previous one and grows exponentially. The increase in new cases of infection in absolute numbers is therefore initially rather small and grows more strongly as the event progresses.

This dynamic development can be dampened if the number of infectious contacts is limited - for example through quarantine or a change in social behaviour with distancing and appropriate hygiene measures - and the number of second infections transmitted per case decreases. With a net reproduction number Rt ≤ 1, the number of new cases of disease no longer increases. If an epidemic cannot be contained during its course, this only occurs after the disease has spread so far in the population that the proportion of susceptible individuals who are not yet infected is greatly reduced. As a result, the number of new infections continues to decrease after some time until the disease reaches endemic status or dies out in the population (population dynamics).

Recording the increased occurrence of new cases of disease as early as possible is essential for the protection of the population. Many affected people search the internet for information on diseases. The evaluation of data from search engines can therefore provide indications for the early detection of epidemics. The evaluation of personal news services on the Internet can also be used for this assessment. However, a frequent search for a disease or its mention on the internet is not necessarily always the result of an increased prevalence or incidence of that disease. Therefore, over-predictions may be made unless other additional data sources are included in the assessment.

Epidemic diseases include various tropical diseases such as dengue fever, but also cholera, influenza, typhoid fever and poliomyelitis (paralysis in children). In the past, anthrax epidemics occurred more often in the downstream area of tanneries. Probably the most devastating epidemics in human history were caused by the plague. One of the most devastating pandemics in world history is called the Black Death, which claimed an estimated 25 million lives in Europe between 1346 and 1353 - a third of the population at the time. The cause is believed to be the plague caused by the bacterium Yersinia pestis. The word "plague" is derived from the Latin word pestis, meaning pestilence, and is therefore also used without direct reference to the disease.

reference to the disease. According to current knowledge, the pandemic first occurred in Central Asia and reached Europe via trade routes (including the Silk Road). From the eastern Mediterranean region, the disease probably spread to the rest of Europe via rat fleas, but some areas were relatively spared. In what is now Germany, it is estimated that one in ten inhabitants lost their lives as a result of the Black Death. Bremen, Hamburg, Cologne and Nuremberg were among the cities in which a very high proportion of the population died. In contrast, the number of deaths in the eastern part of present-day Germany was much lower.

Figure 3.10: A woman suffering from the plague

In his work Decamerone, the contemporary witness Boccaccio impressively described how, after the outbreak of the pandemic, many inhabitants of Florence no longer fulfilled their social obligations:

"We will keep silent about the fact that one citizen avoided the other, that almost no neighbour cared for the other and even relatives did not see each other at all or only rarely and then only from afar. The terrible visitation had created such confusion in the hearts of men and women that one brother left another, the uncle left the nephew, the sister left the brother and often the wife left the husband; yes, what seems even stranger and almost unbelievable: father and mother shied away from looking after their children and caring for them - as if they were not their own. Many died who, if they had been looked after, would probably have recovered. But because of the lack of proper care necessary for the sick, and because of the power of the plague, the number of those who died day and night was so great that it was horrifying to hear about it, let alone witness it."

Many of the people who saw the Black Death as God's punishment found solace in religion at this time. Religious movements arose spontaneously in the entourage or in anticipation of the plague - many of these challenged the church's monopoly on spiritual guidance. Prayer services and processions marked everyday life. Flagellants paraded through the towns in "Geißlerzüge". The "plague saint" St. Roch was intensively venerated, pilgrimages increased. In many places, churches and other monuments such as so-called plague columns bear witness to people's fear and their desire for deliverance from the plague. Various people tried to make the most of every minute of their lives, and with dance and music they tried to escape the Black Death. The Italian chronicler Matteo Villani wrote:

"The people, realising that they were few and had become rich through inheritance and the giving away of earthly things, and forgetting the past as if it had never been, were more dissolute and wretched than ever before. They gave themselves up to idleness, and their disruption led them into the sin of gluttony, into carousing, into taverns, to delicious food and gambling. They threw themselves into the arms of lust without a second thought." A functioning economy could no longer be maintained under the impact of a pandemic. Workers died, fled and no longer performed their duties. For many, it seemed pointless to cultivate the fields when death would soon overtake them.

Ecclesiastical and secular power rapidly lost authority in the face of the helplessness with which they met the pandemic. The poet Boccaccio noted in his Decamerone: "In such lamentation and affliction of the city, even the venerable prestige of the divine and human laws had almost sunk and been destroyed; for their ministers and executors, like the rest of the inhabitants, were all sick or dead, or had retained so few assistants that they could no longer perform official acts. Therefore, everyone could get away with whatever he wanted."

The people who suffered most from the loss of authority of the secular and ecclesiastical power were those who belonged to the cultural fringe groups of the medieval societies. Thus, in the course of the pandemic, there were severe pogroms of Jews, which the ecclesiastical and secular rulers could no longer stop and which resulted in only a few Jews living in Germany and the Netherlands after 1353. The pogroms broke out because the angry people believed that the Jews were the culprits for the catastrophe. In view of the plague epidemic, a physician from Breslau named Dr. Heinrich Rybbinus stated that the people, in their perplexity, had fallen to burning Jews and fighting each other. The first attacks against Jews began in Toulon on Palm Sunday 1348: shortly after the first plague deaths occurred there, parts of the city population attacked the Jewish quarter and killed 40 people. A few days later, there were also attacks in Avignon, Grasse and other cities in Provence and then in Catalonia.

The rumour that certain groups of people dripped poison into wells and springs circulated very frequently in times of need and was, for example, reproached to lepers in 1321 after the Shepherds' Crusade of 1320. Very soon after the first plague deaths, this was also reproached to Jewish fellow citizens: In Savoy, Jewish defendants had pleaded guilty to such offences under torture. Their confession quickly spread throughout Europe and was the basis for a wave of attacks in Switzerland and Germany - especially in Alsace and along the Rhine. On 9 January 1349, part of the Jewish population was murdered in Basel - the Basel city councillors had previously banned the worst agitators from the city, but under pressure from the city population they had to lift this ban again and expel the Jews instead. Some of the expellees were arrested and burned in a house built especially for them on an island in the Rhine. In Strasbourg, the city government also tried to protect the resident Jews, but was removed from office with the votes of the guilds. The new Strasbourg city government tolerated the subsequent massacre, which in February 1349 - i.e. at a time when the Black Death had not yet reached the city - claimed the lives of 900 out of 1,884 Jews living in Strasbourg. In March 1349, four hundred members of the Jewish community of Worms burned themselves to death in their homes to escape compulsory baptism; in July 1349, the Jewish community of Frankfurt also committed suicide in this way. In Mainz, Jews resorted to self-defence and killed 200 attacking townspeople. Even the Jewish community living in Mainz - the largest in Europe at the time - ultimately committed suicide by setting fire to their own houses. The pogroms continued until the end of 1349. The last ones took place in Antwerp and Brussels. For cities such as Freiburg im Breisgau, Cologne, Augsburg, Nuremberg, Königsberg and Regensburg, it is assumed that even before the local outbreak of the plague, flagellants incited parts of the population to murder the Jewish population as well poisoners. However, more recent research assumes that shifting the blame onto the flagellants is mostly just a convenient attempt by 14th century historians to justify the murders. In addition to the search for a scapegoat and a since the

As the intolerance of the church towards those of other faiths increased in the 12th century, greed was also an important motive for the murder of Jewish citizens. The importance of the Jews as money lenders was no longer as great as it had been in the 12th century.

The murder of the Jews was not a crime of the 13th century, but apparently a large part of the population also saw the murder of the Jews as a way to get rid of their creditors. The mayor of Augsburg, Heinrich Portner, was heavily indebted to Jewish money lenders and willingly allowed the Jews to be murdered.

There was no lack of people who drew attention to the injustice of these murders. As early as 4 July 1348, Pope Clement VI, who lived in Avignon, issued a bull against the persecution of Jews. The papal bull was only effective in Avignon and otherwise did relatively little to protect the Jews. Therefore it was followed on

26 September 1348 a second papal bull with the title Quamvis perfidiam. In it, he described the accusation that the Jews were spreading the plague by poisoning wells as inconceivable, since it raged in areas of the earth where no Jews lived, and where they did live, they themselves became victims of the plague. He called on the clergy to place the Jews under their protection. Clement
VI - who himself collected Hebrew manuscripts - also forbade killing Jews without trial or plundering them. He threatened the persecutors with excommunication. He declared the gangs of Geißler, who had been particularly prominent in the pogroms against the Jews, to be heretics. Similarly ineffective were the measures taken by Queen Joan I of Naples, who in May 1348 reduced the tax burden of the Jews living in her Provençal domain by half to take account of the looting. In June of the same year, their officials were expelled from the Provençal towns, illustrating the Jews' lack of protection due to the progressive loss of authority by the rulers. Like Pope Clement, Peter IV of Aragon, Albrecht II of Austria and Casimir III of Poland were staunch protectors of their Jewish inhabitants. Even if they could not completely prevent acts of violence, massacres such as those in Brussels and Basel did not occur. Casimir III also offered the Jews the opportunity to settle in his territory. An emigration of mainly German Jews to Poland began, which lasted until the 16th century. Casimir III saw the settlement of Jewish citizens as an opportunity to increase the size of the population, which had been decimated by the Mongol raids, and thus to further develop his country economically.

On the other hand, there was no lack of secular rulers who took advantage of the so-called plague pogroms. The Roman-German King Charles IV was at least guilty of complicity: to pay off his debts, Charles pledged the royal Jewish registry, including to Frankfurt am Main. It was even stipulated what was to happen to the property of Jews if "the Jews there were soon to be slain" (Frankfurt documents of 23, 25, 27 and 28 June 1349, relating to Nuremberg, Rothenburg ob der Tauber and Frankfurt am Main). Although he was able to effectively protect the Jews in his domain, this event raises many questions about Charles' character, especially since Charles otherwise always strove to present the image of a just Christian ruler. In fact, the toleration of the murders also violated the legal understanding of the time, since the Jews were under the direct protection of the king and also made payments for this. The Margrave of Meissen went even further, calling on the city population of Meissen to attack Jews at the beginning of 1349 and assuring them that no sanctions would follow such attacks.

In the long term, the plague brought about and accelerated a profound change in medieval European society. As David Herlihy shows, the generations after 1348 did not simply maintain the social and cultural patterns of the 13th century. The massive population collapse caused a restructuring of society that had a positive effect in the long term. Thus Her- lihy described the pandemic as "the hour of the new men": depopulation gave a larger percentage of the population access to farms and rewarding jobs. Marginal lands that had become unprofitable were abandoned, which in some regions led to villages being abandoned or not repopulated (so-called deserts), and the forests that had been cleared in the High Middle Ages in the course of land expansion spread again. The guilds now also admitted members who had previously been refused admission. While the market for agricultural rents collapsed, wages in the cities rose significantly. This meant that a greater number of people could afford a higher standard of living than ever before; however, there were also food shortages in some places because many fields were no longer cultivated, for example in England, where wages for agricultural labourers rose sharply. Although the nobles enforced the Statute of Labourers in Parliament in 1349, which limited wages for field work, farm labourers were also paid in kind. The wage disputes eventually led to the Great Peasant Revolt of 1381, as a result of which England became the first country in Europe to abolish serfdom. Free peasants were subsequently replaced by tenants, and less labour-intensive sheep farming displaced arable farming.

The population reached a low point in Europe around 1400. The significant rise in labour costs ensured that manual work became increasingly mechanised. This made the late Middle Ages a time of impressive technical achievements. David Herlihy cites the printing of books as an example: as long as the wages of scribes were low, the hand-written copying of books was a satisfactory method of reproduction. As wages rose, extensive technical experimentation began, which ultimately led to the invention of letterpress printing with movable type by Johannes Gutenberg.

In Italy, especially in Tuscany, the plague weakened the aristocracy and led to the rise of the artisans and small merchants, who were often debtors of the Jews. These often lost the protection of the princes. Land became relatively devalued as it was now abundant, while capital gained importance and could be accumulated more easily.

The Church - heir to numerous plague victims - emerged from the Black Death richer but more unpopular. Neither had it found a satisfactory answer to the question of why God had imposed such a trial on humanity, nor had it provided spiritual succour when people's need for it was greatest. The movement of the flagellants had tested the au-tority of the Church. Even after this movement had subsided many sought God in mystical sects and reform movements, which ultimately caused the Catholic unity of faith to break apart.

In particular, the Austrian cultural historian Egon Friedell argued in his work Kulturgeschichte der Neuzeit (Cultural History of the Modern Era) that the plague of 1348/49 had caused the crisis of the medieval view of the world and of man and had shaken certainties of faith that had existed until then. He sees a direct, causal connection between the Black Death catastrophe and the Renaissance.

The first major pandemic, which went down in the history books as the Black Death, ended in 1353. In the following years, it flared up again and again in individual regions of Europe as the epidemic became endemic: In local and regional epidemics, it occurred at almost regular intervals in various parts of Europe over the next three centuries, for example in 1400 as the second worst epidemic of the late Middle Ages or the young modern era. The Great Plague of London in 1665/1666, which killed about 100,000 people in southern England (70,000 of them in London alone) or the Great Plague of 1708 to 1714 in northern and eastern Europe with about one million deaths. At the end of the 19th century, the third plague pandemic began in China.

Figure 3.11: A medieval face mask to protect against infections

Most works of art dealing with the effects of the Black Death were written after the pandemic years of 1347 to 1353, with the exception of "Il Decamerone" by Giovanni Boccaccio, which, as far as we know today, was written between 1350 and 1353. The location of the frame story is a country house in the hills of Florence, two miles from the then city centre of Florence. It is to this country house that seven girls and three young men fled from the Black Death that struck Florence in the spring and summer of 1348. The introduction to the book is one of the most detailed medieval sources on the impact of the Black Death in a city.

The Black Death also became an important theme in the art of the late Middle Ages. Artists such as the Lübeck painter and carver Bernt Notke impressively depicted the event in the form of the Dance of Death, which was also incorporated into music. The Black Death was also used in the Peasants' War panorama by Werner Tübke. It was symbolised there by a large open coffin with the terminally ill in the scene `The Plague Sick`. The management of these problems continued into the 20th century. Very impressive is the novel
"The Plague" by the French writer Albert Camus from 1947.

The 2014 to 2016 Ebola fever epidemic in West Africa and the 2018 to 2020 Ebola fever epidemic in the east of the Democratic Republic of Congo are also examples of epidemic outbreaks of Ebola fever in terms of case numbers and timing. In the case of influenza, one speaks of an `influenza wave` when significant proportions of the population are infected in different regions during a season (Fig. 3.12)

Figure 3.12: An Ebola treatment centre in West Africa

Epidemics can be characterised according to spatial and temporal characteristics of the event as well as according to the conditions of occurrence and spread. The following types of epidemics can be distinguished:

- `Point source epidemic`: An epidemic whose pathogens have spread briefly and simultaneously from a point source.
- `Small-space epidemic`: An accumulation of incidences in a spatially limited milieu, e.g. in a home, a children's facility or a school.

- `Scattered epidemic`: Increased occurrence of infections in different places with a common cause, e.g. spread by population movements or food transports.

- `Explosive epidemic`: An explosive epidemic is an epidemic with a sudden increase in the number of cases. Epidemics of this type are often associated with certain transmission factors, for example as infections transmitted via food or drinking water.

- `Mixed epidemic`: A mixture of an `explosive epidemic` and a tardive epidemic, in which the infectious event is initially explosive and a tardive epidemic develops in the course.

- `Tardive epidemic` is the name given to an epidemic with slowly but steadily increasing numbers of cases. Along with the explosive epidemic, the tardive epidemic is one of the two classic basic types of epidemic. In contrast to the term contact epidemic, the term Tardive epidemic, similar to the explosive epidemic, is intended to characterise the temporal course of an epidemic; the mode of transmission is largely irrelevant here. Tardive epidemics can be caused, among other things, by long incubation periods, by infection routes that only lead to the infection of a single person (sexually transmitted diseases), by a small number of carriers, by a strongly developed immunity (e.g. as a result of previous latent or manifest infection) or by the interaction of various factors. Diseases that can lead to a typical Tardive epidemic include plague, smallpox, influenza and HIV.

- `Contact epidemic`: In this type of epidemic, infections and diseases increase through direct person-to-person contact, e.g. through droplet infections, contact infections, or sexually transmitted diseases.

- `Shipping epidemic`: Due to the shipping of contaminated food, there is an increase in foodborne infections in various locations.

- Epidemic via a generally accessible medium: An epidemic whose disease triggers have spread via a generally accessible medium such as air, drinking water or food.

- `Graft epidemic`: An epidemic arising from an endemic.

- `Provocation epidemic`: An epidemic that has arisen following the activation of latent infections as a result of a reduction in resistance in the population.

- Summative epidemic: An epidemic that arises from an endemic situation through a summation of infections (compression wave, attraction wave) because susceptible individuals have accumulated and a pathogen with high contagiousness is spreading.

- Complex epidemic: An epidemic induced by several disease triggers.

- A pseudo-epidemic is a locally increased occurrence of cases of an infectious disease that is due to an increased manifestation of infections caused by a sudden increase in susceptibility in the population or occurs due to increased diagnostic activities and is not triggered by a real increase in new infections.

Epidemics have been dealt with in different ways throughout history. In the case of leprosy in the Middle Ages, the respective government reacted with banishment, exclusion and exclusion of the sick; in the case of plague in the early modern era, it developed strategies of surveillance and confinement, then also disciplinary mechanisms, control networks and meticulous observation of individuals; in the case of smallpox from the end of the 18th century, it reacted with vaccination measures, immunity strategies, statistical surveys and risk assessments. In the face of the AIDS epidemic, homosexuals were initially persecuted and risk groups denounced; later, the handling of the epidemic shifted more to areas outside the closed field of medical observation.

Smallpox was completely unknown in America, the whites had brought it with them from Europe, the native population was completely defenceless. About 56 million Indians died of this disease.

The Spanish flu (40-50 million deaths) occurred at the end of World War 1 and also had very severe consequences. The US Centres for Disease Control and Prevention describe the 1918 flu pandemic as the most severe pandemic in recent history. The first cases were reported in March 1918 at a military base at Fort Riley, Kansas, but the disease became known as Spanish flu because it was first described in the Spanish press. The lack of effective vaccines or treatments (the first influenza vaccine was not developed until the 1940s), World War I (travelling soldiers brought the disease with them) and a shortage of health per- sonals made dealing with the pandemic difficult. In the US, local responses resembled today's COVID-19 control measures: "Officials imposed quarantines, ordered citizens to wear masks and closed public places.

Figure 3.13: An Indian woman suffering from smallpox

including schools, churches and theatres. People were advised to avoid shaking hands and stay at home. Libraries stopped lending books and regulations were issued banning spitting."

In 1981, the medical profession was baffled by the appearance of rare forms of pneumonia and skin cancer in young homosexual men in the USA. In the course of the following year, these diseases also appeared in drug users and recipients of blood transfusions. French scientists were the first to isolate the human immunodeficiency virus (HIV), which causes acquired immune deficiency syndrome (AIDS). It is believed that the virus was transmitted to humans from chimpanzees in Central Africa in the early 20th century and eventually spread to all continents. There is still no cure for AIDS. Although the development of antiretroviral drugs from the mid-1990s onwards has now made AIDS a manageable chronic disease for many patients, especially in wealthier countries, almost 700,000 people still died from AIDS-related illnesses in 2019.

The third plague pandemic of 1855 caused 12 million deaths. It originated in China in 1855 and later spread to India and Hong Kong. At least in the beginning, it was mainly spread by fleas. This pandemic was considered active until 1960, i.e. until the number of cases fell below a few hundred infected persons.

lay. Today, bubonic plague is still a public health threat in some countries, especially in remote areas of Madagascar, where small outbreaks frequently recur. The disease, which once claimed the lives of millions and threatened the survival of nations, can now be easily treated with antibiotics.

In 1545, a mysterious disease called Cocoliztli appeared among the indigenous people of Mexico. Between 5 and 15 million deaths are attributed to it. The cause of the disease has never been clearly identified, although DNA taken from the victims' teeth has revealed a surprising new possible causative agent. Dental pulp - the soft, living tissue in the teeth - is full of blood vessels and therefore with all the pathogens that once circulated in the blood. The outer hard enamel protects the DNA of these pathogens for centuries. Scientists concluded that Salmonella enterica, a bacterium that spreads through contaminated food or water, was present in many of the teeth, strongly suggesting that this bacterium was responsible for the disease, although it is by no means the only possible cause of the disease. A similar epidemic hit Mexico in 1576 (Fig. 3.14).

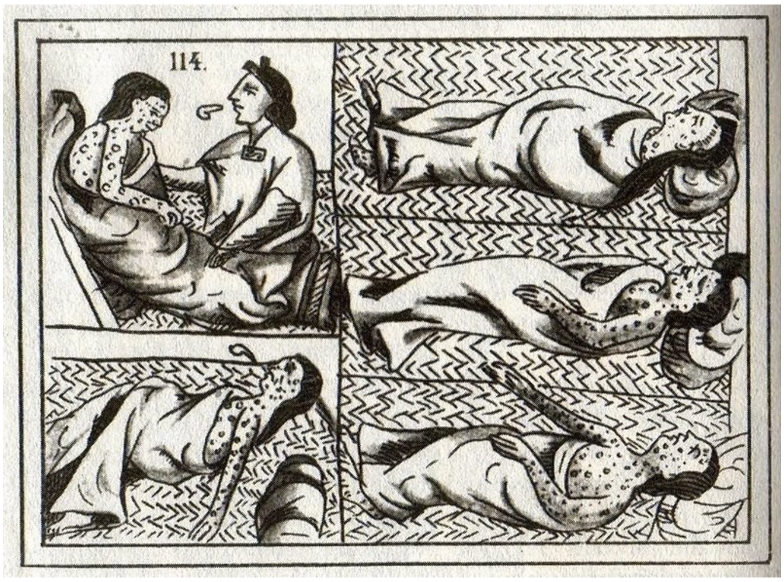


Figure 3.14: Victims of the Cocoliztli epidemic in Mexico

The Antonine Plague was a smallpox epidemic that struck much of the Roman Empire around 165 AD, killing up to 10 per cent of the population. According to Smithsonian Magazine, at the height of the plague in 189, the following died
n. The plague raged so devastatingly within the professional armies of the empire that attacks were broken off. It decimated the aristocracy, and abandoned farms and depopulated cities marked the landscape of Egypt

to Germany." The Romans, however, found ingenious ways to maintain the structures of society, including making freed slaves eligible for local office, and the empire continued for another two centuries. The Roman historian Cassius Dio believed that "the trauma of experiencing the plague can be overcome if a well-governed society works together to recover and rebuild itself, and the society that emerges from these efforts can become stronger than ever."

The Spanish flu and the war were not the only potentially lethal threats soldiers faced in the First World War. Between 1918 and 1922, typhus, a bacterial disease transmitted by clothes lice, was spread from Serbia through Russia and the surrounding countries - the Eastern Front of the First World War - resulting in an estimated 30,000,000 infected and
3,000,000 deaths. Soldiers lived in cramped quarters, huddled together for warmth, making it easy for lice to move from man to man, living comfortably in the seams of uniforms and enjoying blood meals.

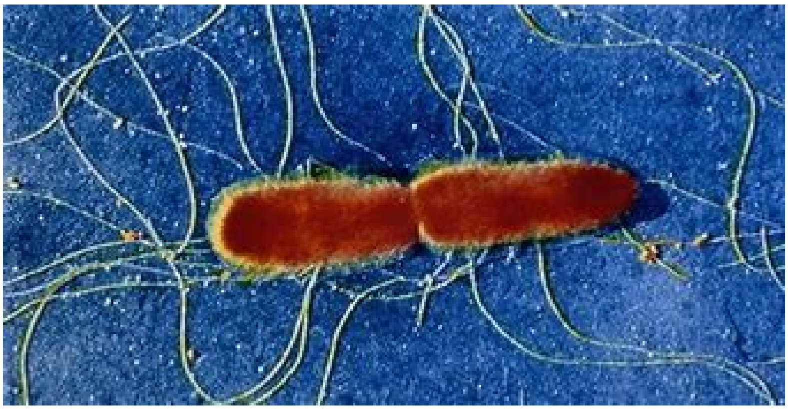


Figure 3.15: Typhoid agent

Of course, COVID-19 (at least 2 million deaths) is currently moving the world, although the figures so far show that the scale of the problem is nowhere near the problems that had to be overcome in the past. At
31 December 2019, health authorities in the Chinese city of Wuhan reported dozens of cases of pneumonia due to an unknown cause. A week later, the cause of the outbreak was identified as a new coronavirus. On 20 January 2020, the first cases outside China were confirmed. By March, the disease had spread around the world, major international events were cancelled, and countries closed their borders to non-citizens and several nations imposed closures. The World Health
organisation estimates that 1.6 billion people are at risk of losing their livelihoods due to the economic impact of the pandemic. By

At this point, the World Health Organisation has reported 2 million deaths from the coronavirus worldwide, over 400,000 in the US alone. The development of several vaccines has raised hopes that the pandemic could be under control in some countries by the end of this year, but the way we travel, work and socialise may never be the same again.

Further epidemics on a larger scale:

- the Russian flu of 1889-90 (1 million deaths)

  In the autumn of 1889, according to the History Channel, an influenza pandemic broke out, powerfully infecting half the population in the Russian capital of St Petersburg. Within a few months, it spread to much of the world via main roads, rivers and railways. It is often considered the first modern pandemic - the first in the age of rail and steamship travel. As long as people could move easily from city to city and country to country, it was almost impossible to stop the spread.

- The Hong Kong flu of 1968 (1-4 million deaths)

  The 1968 influenza pandemic was caused by a highly contagious strain of influenza that first appeared in Hong Kong in the summer of 1968. Within two weeks of the virus strain appearing, almost 500,000 cases were reported. Soldiers returning from Vietnam spread the disease in the United States. The highest mortality rates were among the groups most susceptible to infection, namely infants and the elderly. Although a vaccine against the virus was developed, it was only available after the pandemic had already reached its peak in many countries. The strain of virus that caused the 1968 pandemic is still circulating today.

- Japanese smallpox epidemic (1 million deaths)

  Smallpox first appeared in Japan in 735 AD. It is believed that this epidemic claimed the lives of up to a third of the country's population. Like the smallpox epidemic that struck Central America centuries later, this epidemic was one that struck a population that had not previously been exposed to smallpox. After the first epidemic, outbreaks occurred again every 10 to 15 years for several centuries.

On average, three out of ten infected people died. Smallpox eventually became an endemic disease in most of the country. Mass vaccination campaigns that began in the late 1950s led to the eradication of the disease, in Asia in 1975 and around the world in 1980.

- Encephalitis lethargica (up to 400,000 deaths)

  A mysterious pandemic of sleeping sickness, also known as encephalitis lethargica or encephalitis epidemica, led to up to one million cases worldwide between 1916 and 1926. The disease originated in the cold, damp plains and trenches of northern France and Belgium, the battlefields of the First World War. The disease manifested itself through flu-like symptoms, uncontrolled eye movements and later an irresistible need to sleep. This lethargy lasts for weeks, in some cases even a year or longer. The mortality rate in this acute phase of the disease was 30 to 40 percent. Many of the survivors went on to develop Parkinson's disease. The epidemic disappeared about 10 years later, at the same rate at which it broke out. Scientists have not been able to identify a single cause for the disease, so the causative agent remains unknown and may still exist today.

Figure 3.16: The causative agent of sleeping sickness

- Yellow fever epidemic (100,000-150,000 deaths) Yellow fever is a mosquito-borne virus with a mortality rate of up to 50 percent. A series of epidemics between 1793 and 1905 in the southern and eastern United States claimed thousands of lives. By all accounts, death from yellow fever was extremely unpleasant: victims had a variety of unpleasant symptoms: Jaundice, chills, nausea, headaches,

> Fever, convulsions, delirium. In the later stages of the disease, victims bled through their eyes, noses and ears. People who survived the disease were immune. In 19th century New Orleans, survivors were called acclimatised, according to NPR, and non-acclimatised people had difficulty finding work and housing. Starting in the early 20th century, a series of vaccines were developed, but several vaccine researchers accidentally contracted the disease and died. Yellow fever still claims about 30,000 lives a year, mainly in Africa. Proof of vaccination is required for international travel to many African countries.

One can see from these historical facts that the current Corvid 19 situation is very unpleasant, but by no means new. In terms of fractals, spread is always due to the fact that a point where pathogens exist has contact with a neighbouring point and the pathogen is transmitted with a certain probability, which can be very high. Therefore, the maximum reduction of contacts with other candidates for infection is the most important thing. In today's language: testing and quarantine are the most important elements of control, they prevent transmission to other people and enable treatment of the disease reduced to one person. Vaccination protects the vaccinated person, but can mask the existence of infection and must be supplemented by testing. Unrestricted contacts of vaccinated persons with any other persons are quite risky without testing.

The problem of epidemics is not limited to humans. Best known is avian influenza, which first appeared in Scotland in 1959 and then spread to Europe, Asia and parts of Africa. A number of effective inactivated vaccines against influenza viruses have long been available for animals, and several live vaccines have been developed to market maturity since 2005. In Germany, however, vaccination against avian influenza is only permitted in exceptional cases. There is also an approved DNA vaccine for chickens.

The World Health Organisation has also repeatedly warned against vaccination, as vaccinated animals can no longer be distinguished from virus-carrying animals. Moreover, vaccinated, infected birds could become carriers of the viruses without showing symptoms themselves. There is also the danger that the viruses mutate in undetected infected animals and that these genetic changes could spread more easily than in non-vaccinated flocks, as these are killed after each outbreak; such a development was observed after an H5N2 outbreak in Mexico.

Another example from the animal kingdom is swine fever. Classical swine fever (CSF) has been known as an infectious disease since 1833. This viral infection occurs worldwide, with the exception of North America, Australia, New Zealand and parts of Europe and South America. It is one of the most dangerous pig diseases of all and is still difficult to control and has not been eradicated. Classical swine fever must be distinguished from African swine fever (ASF). Despite the similar symptoms, the ASF and CSF pathogens are not closely related.

The pig plague pathogen is pestivirus C, also called classical swine fever virus. Although it is related to pathogens of other diseases, no other animal species is susceptible.

Wild boar can be considered a permanent reservoir of pathogens. In addition to contact with wild pigs, the purchase of already infected but not visibly ill pigs is often a source of infection. In addition, the virus can also be transmitted by so-called vectors (contaminated vehicles and equipment, clothing or food waste). Infection within a herd then occurs directly from animal to animal, mainly perorally or via the respiratory tract. The incubation period depends on the virulence of the respective pathogen. It can be between 2 days and over 5 weeks. The virus first multiplies in the tonsils and the lymph nodes of the throat. The pathogen is already in the bloodstream after 24 hours and reaches its maximum concentration within a week. If the pathogen is in the blood, it is constantly excreted through urine, saliva, faeces, eye and nose secretions. This is also the reason for the epidemic spread of the disease; large pig herds with several thousand animals can be completely infected within a week.

African swine fever is a separate development that should not be confused with European swine fever.

Here, too, a complete separation of populations is the surest means of preventing further contagions. Consider, for example, the barrier fences erected in eastern Germany to prevent the immigration of wild boar from Poland.

3.10 Digitalisation and Artificial Intelligence

All information obtained in the course of investigations is currently available in digital form. It can be viewed in any size, and that not only once, but very often. They can also be forwarded so that critical problems can be assessed by several specialists. Image processing offers many possibilities to enlarge, rotate etc. the images. In addition, they can be compared with other images. Matching them with images of diseased or healthy people can significantly support the diagnosis.

In addition, there is the development over time. Often you have the development of a person over a longer period of time at your disposal, for example if a person regularly goes for check-ups. If a person falls ill at a certain point in time, the available images can be checked and examined to see when changes become visible. These images can then be specially stored and compared with images of other people, which in turn supports a timely diagnosis and the corresponding measures.

In the last ten years, great success has been achieved in training neural networks. The term "neural network" describes structures in the human brain. Nerve cells (neurons) are connected to each other by means of synapses and thus form nerve networks (neural networks). The neurons form the nodes of the network. Neurons receive stimuli from outside the network or from other neurons. Stimuli can be sensory perceptions, for example.

A neuron usually has one input and branches into several outputs. If the strength of the incoming stimulus exceeds a certain level, the neuron "fires", i.e. it transmits the stimulus to other neurons via synapses. The transmission can take place through electrical impulses or as a chemical reaction with the help of messenger substances. This is how stimuli and information are transmitted and processed in the brain. This is just as important for processing sensory perception as it is for learning processes; neural networks can learn complicated patterns. In doing so, it is not necessary for them to understand the abstract rules behind the patterns.

This structure, which exists in nature, is now reproduced in a model by an artificial neural network. The information is received by the input neurons and output by the output neurons. The hidden neurons lie in between and represent internal information patterns. The neurons are connected to each other via so-called edges. The stronger the connection, the greater the influence on the other neuron.

The following structure is used:

- Input layer: The input layer supplies the neural network with the necessary information. The input neurons process the input data and pass it on weighted to the next layer.

- Hidden layer: The hidden layer is located between the input layer and the output layer. While the input and output layers consist of only one layer, there can be any number of layers of neurons in the hidden layer. Here, the received information is weighted again and passed on from neuron to neuron to the output layer. The weighting takes place in each level of the hidden layer. However, the exact processing of the information is not visible. This is where the name hidden layer comes from. While the incoming and outgoing data are visible in the input and output layer, the inner area of the neural network is basically a black box.

- Output layer: The output layer is the last layer and connects directly to the last level of the hidden layer. The output neurons contain the resulting decision, which emerges as an information flow. Deep learning is a main function of an Artificial Neural Network and works as follows: Given an existing network structure, each neu- ron is assigned a random initial weight. Then the input data is given into the network and weighted by each neuron with its individual weight. The result of this calculation is passed on to the next neurons of the next layer or layer, this is also called "activation of the neurons". A calculation of the overall result happens in the output layer.

There are many different architectures of neural networks.

- Perceptron: This is the simplest and oldest neural network. It takes the input parameters, adds them together, applies the activation function and sends the result to the output layer. The result is binary, either 0 or 1, and thus comparable to a yes or no decision. The decision is made by comparing the value of the activation function with a threshold value.

  If the threshold value is exceeded, the result is assigned a 1, whereas 0 is assigned if the value falls below the threshold value. Based on this, further neural networks and activation functions were developed that also make it possible to obtain several outputs with values between 0 and 1. The best known is the sigmoid function, in which case one also speaks of sigmoid neurons.

- Feed forward neural networks: The origin of these neural networks lies in the 1950s. They are characterised by the fact that the layers are permanently connected to the next higher layer. There is no backward directed edges. The training process of a Feed Forward Neural Network (FF) then usually runs like this:

1. all nodes are connected;

2. the activation runs from the input layer to the output layer. There is at least one more layer between the input and output layer. If there are particularly many layers between the input and output layer, we speak of "deep feed forward neural networks".

- `Convolutional neural networks (CNNs)` are artificial neural networks that can work particularly efficiently with 2D or 3D input data. CNNs are used in particular for object detection in images.

  The big difference to classical neural networks is the architecture of CNNs and thus the name "Convolution" or
  "Folding2 can be explained. In CNNs, the hidden layer is based on a sequence of convolution and pooling operations. In convolution, a so-called kernel is pushed over the data and meanwhile a convolution is calculated, which is comparable to a multiplication. The neurons are updated. The subsequent introduction of a pooling layer ensures that the results are simplified. Only the important information is retained afterwards.

  This also ensures that the number of 2D or 3D input data is reduced. If this is continued further and further, the end result in the output layer is a vector of the "fully connected layer". This is particularly important in classification because it contains as many neurons as classes and evaluates the corresponding assignment via a probability.

- `Recurrent Neural Networks (RNN)` add recurrent cells to KNNs, giving neural networks a memory. The first artificial neural network of this kind was the Jordan network, in which each hidden cell received its own output with a fixed delay - one or more iterations. Otherwise, it is comparable to the classical feed forward networks.

Of course, there are many variations - such as passing the state to input nodes, variable delays, etc., but the basic idea remains the same. This type of NNs is used especially when context is important. Because then decisions from past iterations or rehearsals significantly influence the current. However, since recurrent networks have the decisive disadvantage that they become unstable over time, it is now common to use so-called Long Short-Term Memory Units (LSTMs for short). These stabilise the RNN even for dependencies that exist over a longer period of time. The most common example of such dependencies is word processing - a word can only be analysed in conjunction with earlier words or sentences. Another example is the processing of videos, for example in autonomous driving. Objects within image sequences are recognised and tracked over time.

The sigmoid function is often used as the threshold function.

$$f(x) = \frac{1}{1 + e^{-x}} \tag{3.4}$$

is used. For $x \to +\infty$ it tends towards +1, for $x \to -\infty$ it tends towards -1. For $x = 0$ it has the value $\frac{1}{2}$. This function can be used to decide between 0 and 1, that one outputs the value 0 for $f(x) < \frac{1}{2}$, for $f(x) \geq \frac{1}{2}$ the value 1.

A neural network is trained in three steps.

- Step 1: Determine the output for specific data sets. For this, all training data sets are used as input for the model and the calculated results are collected.

- Step 2: Determine the deviation or error of the output from the desired output. With the help of the error function, one determines for each training data set how great the deviation of the actual result from the expected result is. The average error value of all training data sets now forms the basis for the third step.

- Step 3: Backpropagation of the error and adjustment of the weights of the network. In backpropagation, the average value is used to determine the direction of the parameter. In doing so, one must find out to what extent the weights of the network must change in order to minimise the error.

  For this we form the derivative of the error function. This now indicates what the slope of the error function is for the current parameters. If the slope of the error function for a parameter is greater than 0, then we know that the parameter must become smaller. If the slope of the error function is smaller than 0, then the parameter must become larger. To achieve the desired result, the process must now be carried out for each parameter of the model.

With the development of ever more powerful computers, the sizes can of course be greatly increased, for example to the size of a monitor for the evaluation of medical data. For a 27-inch monitor (68.58 cm), you have to consider 1920 x 1080 = 2,073,600 pixels. This means that the input layer must contain 2,073,600 neurons. This order of magnitude naturally allows for very subtle classifications.

4 Fractal structures in human society

We go back to Brownian motion (section 2.9) and use this concept for groupings of people. It is easy to see that there are a lot of fractal structures here. Many interesting explanations of these problems can be found in the book by Elias Canetti: "Masse und Macht".[**18]**

4.1 Groups of people and their behaviour

The appearance of groups of people is an everyday phenomenon that everyone is familiar with. School classes, travel groups, teams of different sports, spectators at competitions - in the sense of a Brownian movement, these are particles that have come together and stay together, with a common goal. It is an open mass, all members have come together voluntarily and also want to remain a part of this mass.

That changes very quickly when a real or perceived danger exists. Reports from the world of sport, especially football, are almost part of everyday life.

"Football has seen many tragedies in the stands. The disaster at the Heysel stadium in Brussels in 1985 was broadcast live in 77 countries. People were often crushed because the exits were locked. And in Moscow, 340 spectators died during a European Cup match. The worst stadium disasters in football (Süddeutsche Zeitung):

- 24 May 1964: Riots break out in the national stadium of the Peruvian capital Lima during an Olympic qualifying match between the football teams of Peru and Argentina after the referee disallows a goal shortly before the end. Because Peru would thus miss out on qualification, riots break out. The police shoot tear gas, whereupon spectators are forced to

Figure 4.1: Panic at the exit of a football stadium

are pushing towards the exits. But the gates are still locked and can only be opened inwards. Sad end: 318 dead, more than 500 injured.

- 17 September 1967: There are 44 dead and 600 injured in Kayseri/Turkey when spectators fight each other with guns, knives and broken bottles after a disputed goal.

- 23 June 1968: 74 people are killed and more than 150 injured in the Argentine capital Bue- nos Aires. Fans had flocked to an exit that was locked after the match between River Plate and Boca Juniors. The mass panic was triggered by the Boca fans. They had thrown burning newspapers into the crowd. The incident was never fully resolved in court. No one has been convicted of violent action.

- 2 January 1971: 66 dead at Glasgow's Ibrox Park when crowds break through railings after the Rangers - Celtic local derby.

- 17 February 1974: Spectators at the Zamalek Cairo vs. Dukla Prague match try to break through the barriers to a stadium in the Egyptian capital Cairo, 49 are trampled to death.

- 20 October 1982: Up to 340 people die in the European Cup match between Spartak Moscow and HFC Haarlem from the Netherlands. The police are said to have forced fans out before the final whistle by making the exits too narrow and icy.

the Moscow stadium. When a late goal was scored, the fans tried to get back into the stadium and mass panic ensued. The Russian authorities officially speak of 61 dead and see no blame on the police.

- 1 May 1985: A cigarette sets fire to the wooden grandstand in Bradford, England, killing 56 people and injuring hundreds.

- 29 May 1985: During the European Champion Clubs' Cup final between Juventus Turin and Liverpool FC, English fans stormed a block of Italian fans in the Heysel Stadium in Brussels, panic broke out, 39 people died and 400 spectators were injured. The deaths were broadcast live in 77 countries. The barrier between a block of Liverpool and Juve fans consisted of a chicken fence and eight policemen. After the England attack, the Italians fled downstairs where a boundary wall burst. The horror pictures went around the world, yet the final kicked off. Juventus won 1:0.

- 12 March 1988: At least 93 people are killed in a mass panic in Kathmandu (Nepal). They had tried to get to safety from a hailstorm, but locked stadium doors became their prison.

- 15 April 1989: At the Nottingham Forest v Liverpool FC match at the Hillsborough Stadium in Sheffield, fans try to force their way into an already overcrowded stand, people are crushed against the boundary fences, 95 people are killed, more than 200 injured.

- 13 January 1991: In Orkney (South Africa) 40 people die in riots, most of them are trampled or crushed to death.

- 5 May 1992: 15 people killed and around 1,300 injured when an auxiliary stand collapsed shortly before the start of the French Cup semi-final between SSC Bastia and Olympique Marseille.

- 16 October 1996: During the World Cup qualifying match between Guatemala and Costa Rica, 78 people are killed in a mass panic in Guatemala City.

- 11 April 2001: 47 people are killed in Johannesburg, South Africa, during the match between the Kaizer Chiefs and the Orlando Pirates. Fans had tried to get into the crowded stadium and had been pushed into barbed wire.

- 9 May 2001: In Ghana's capital Accra, frustrated fans throw dismantled stadium seats and bottles onto the pitch after the Accra Hearts of Oak's championship match against Kumasi Ashanti Kotoko. The police used tear gas and closed the stadium gates. As a result, panic breaks out among the 40,000 spectators, 124 are trampled to death or suffocate, and more than 150 people are injured.

Only the two fires pose a real danger, the other victims are due to the mindless behaviour of the spectators.

Demonstrations also often lead to violent confrontations, either between hostile groups or between demonstrators and the police. The individual particles lose their individuality and drift in a herd.

4.2 The communication behaviour of groups

The number of members in a group is not very high, ten to fifteen should be the normal values. If a group is newly formed, then the group members start to communicate.

Unresolved conflicts, a wrong expectation or current difficulties flow into the communication. Firm rules of conversation help to reflect on one's own behaviour and to adopt a self-critical attitude. The following rules should be observed:

- Do not give unsolicited advice.

  Depending on the situation and the person, unsolicited advice can cause defiance. You can have the best of intentions, but quite a few people feel patronised and not taken seriously enough when they receive unsolicited feedback. One possible rule of conversation would be to ask the other person: Would you like to hear my opinion on this? I can think of a few thoughts spontaneously - would you be interested to know what another possibility would be?

- One should not draw conclusions from oneself to the general public.

  The point of discussions is not to bring out the egocentric, but you don't have to hide either.

- You have to let other people finish.

  Probably the most important rule of conversation is not to interrupt other people immediately, but to let them finish. This is not always easy, especially when a statement is made that makes you feel directly attacked. Nevertheless, it is important that other people get the chance to state their point of view - not everyone can express themselves in an incredibly pointed and polished way. Some critical passages clarify themselves once the interlocutor has had a chance to complete his or her sentence.

- Avoid monologues.

  Some people tend to monologue. This is not necessarily malice, but sometimes a lack of precision. It is better to try to concentrate on the core message and not get lost in side issues. This way you give other people the opportunity to contribute to the topic or to ask questions. You also prevent the other person from getting tired.

- Announce the need to speak.

  Even if you are no longer in school and don't need to point fingers: Let us know if you need to talk. Maybe the colleague in the meeting accidentally went back to the slide from the previous year and gave the wrong figures? Before new errors develop on the basis of false assumptions, it can make sense to intervene at an early stage. Nevertheless, do so with the greatest possible respect: Mr Müller, if I may interrupt you briefly, it seems to me that data material from the previous year was used here. . .

- Make personal statements instead of asking.

  Questions, especially questions of understanding, are absolutely justified in discussions. A rule of conversation should be to use questions only for that purpose and not to express hidden criticism. Rhetorical questions or packaged accusations do not belong, a negative example: Have you ever thought about the effort that would result? Such questions show the other person up and can come across as condescending. You want to avoid that, so you should rather give direct substantive feedback: I could imagine that, based on our experience last year, the effort will be very great. We were already short-staffed last year, so one possibility would be ...

- Send 'I' messages.

  A classic among the rules of conversation is that you send so-called "I" messages: I have observed that people often forget to refill the toner in the copier. I-messages have the great advantage that your counterpart does not immediately feel attacked if you formulate something critical. You-messages automatically have an accusatory effect and can easily lead to your interlocutor going on the defensive. In the worst case, the fronts harden. The crowning glory is a combination with "always", "all the time" or "never" - this is the only way a conversation can go wrong. Another rule of conversation should be to avoid these words if possible.

- Address others directly.

  You should include present colleagues in the conversation. You can do this by talking to them and not about them. Negative example: Mr Schneider is of the opinion that nothing will happen before the end of the year. Better: Mr Schneider, as I understood you earlier, you also see the time frame. ... .

- Speak only when you have the floor.

  This rule of conversation is actually a matter of course, but sometimes meetings are like kindergartens: If possible, only one person should speak at a time. There are several reasons for this: Firstly, it is impolite to the person who has the floor. Secondly, confusion makes it more difficult to understand and concentrate on the other person. Thirdly: What you or someone else has to say could also be important for the rest of the group. But if everyone is talking at the same time, it gets lost.

- Practice consideration.

  Each group has participants who tend to want to contribute more and others who are much more reserved. Observe your conversation partners: are there perhaps signs that someone wants to say something but does not dare? A rule of conversation is to show consideration, to withdraw yourself at one point or another and to encourage others so that they also have the opportunity to express themselves.

A widespread phenomenon corresponds to the behaviour of Cantor's dust: one whispers to someone mysteriously and confidentially a false or distorted information into the ear. This information often spreads very quickly across many people. With the opposite, a second opinion can be circulated that is diametrically opposed to the first opinion.

A rumour is unconfirmed news that is always of general or public interest, spreads diffusely and whose content is subject to more or less strong changes. The most important characteristic of a rumour is the uncertainty of the information passed on. According to the classification of Robert
H. Knapp (1944), rumours can be divided into three categories:

- Wishful thinking (hope for a positive event)
- Rumours of aggression (hostility towards others)
- Fear rumours (fear of a negative event)

Furthermore, rumours can be classified according to their content, such as organisational rumours (DiFonzo et al., 1994) or product rumours (Miller, 2013).

Whispered propaganda is the term used to describe a process in which events, usually kept secret by politicians, are passed on and thus slowly become known to the population and thus to the public. This dissemination of news, which often occurs in totalitarian states, can lead to rumours.

Latrine slogans are colloquially derogatory rumours that are mostly misleading or false and are spread secretly. The word originates from the language of soldiers, since in barracks or other accommodations all ranks met at the septic tank or latrine there to defecate together and where information was also exchanged and then passed on. Synonyms are latrine rumour or crude shithouse slogan.

The term "Stammtischparolen" refers to stereotypical set pieces of a local opinion and also includes rumours.

In all these phenomena, of course, the self-similarity of communication is easy to see.

Advertising is often close to rumours and slogans. One trusts that an endless repetition of a name for a product will become so fixed in the mind of a customer that he will prefer this product the next time he makes a purchase. Often, however, the opposite is the case. At Amazon, it has been noticed that a customer decides after three to four offers or ends the search. "A lot helps a lot" is not always true. It is even worse when you have bought something online and then this product is offered again and again.

Often people use descriptions in the wording of advertisements that are not confirmed by anything: "cheapest", "most beautiful", "fastest", etc.

5 Fractals in the Nature

It has been shown in the evolution of biological systems that fractal structures are optimal:

a) It succeeds in accommodating the longest possible structure within the smallest possible area.

b) It succeeds in accommodating as large an area as possible within the smallest possible volume.

5.1 Fractals in inanimate nature

Irregular surfaces abound in the real world, from atoms to mountains to outer space. One usually proceeds in this way:

- Description of their geometry,
- Explanation of their existence,
- Understanding of their properties.

The mountains shown in Fig. 5.1 are a typical example of fractal bodies. They are possibly the remains of a volcano that has been gnawed by the ravages of time; erosion, wind, water and different temperatures have caused many small pieces to splinter off, the hardest parts are still there.

Coastlines have become the standard example of fractals.

The ocean waves wash out smaller, softer pieces of rock on a coast first, creating more and more irregularities. This process is called erosion. Small indentations form, which then become larger and larger due to the force of the water. In this way, the

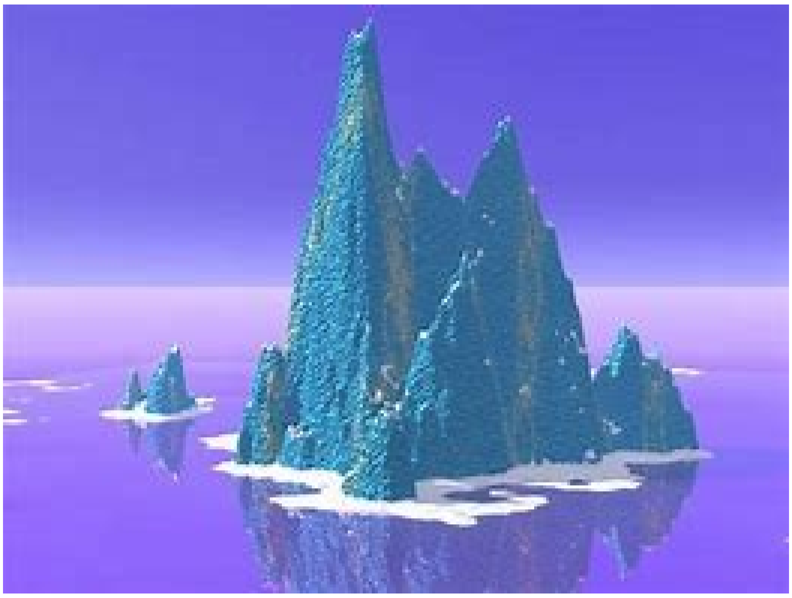


Figure 5.1: Fractal mountains

coast gradually, and the force of the waves is distributed over a longer coastline. Fractal coasts are therefore less exposed to erosion. Geographers have modelled this course of development of coastal morphology. In their model, the strength of the waves is inversely proportional to the length of the coastline. The scientists found
found that this process reaches a stable point when the coast reaches a fractal dimension of exactly 4/3 = 1, 333.

Determining the length of the coastline of England was the starting point of Mandelbrot's work. On a satellite image you can see the fractal structure very nicely.

For further investigations, an observation by Robert Brown should be discussed. In 1827, he discovered that the smallest particles (pollen grains) suspended in a liquid perform disordered zigzag movements. This movement is therefore also called Brownian motion. If you plot the movement of a particle in space and only consider one dimension, you get a function over time. The horizontal axis represents time, the vertical axis represents the state of the movement. The analysis of this function showed that Brownian motion is statistically self-similar and has a fractal dimension of 1.5.

B. Mandelbrot and J. van Ness generalised Brownian motion by defining fractal Brownian motion. With this, they characterised a family of stochastic processes based on the normal distribution. It was now possible to simulate the 1.5-dimensional Brownian motion with any dimension between 1 and 2. During these investigations, B.

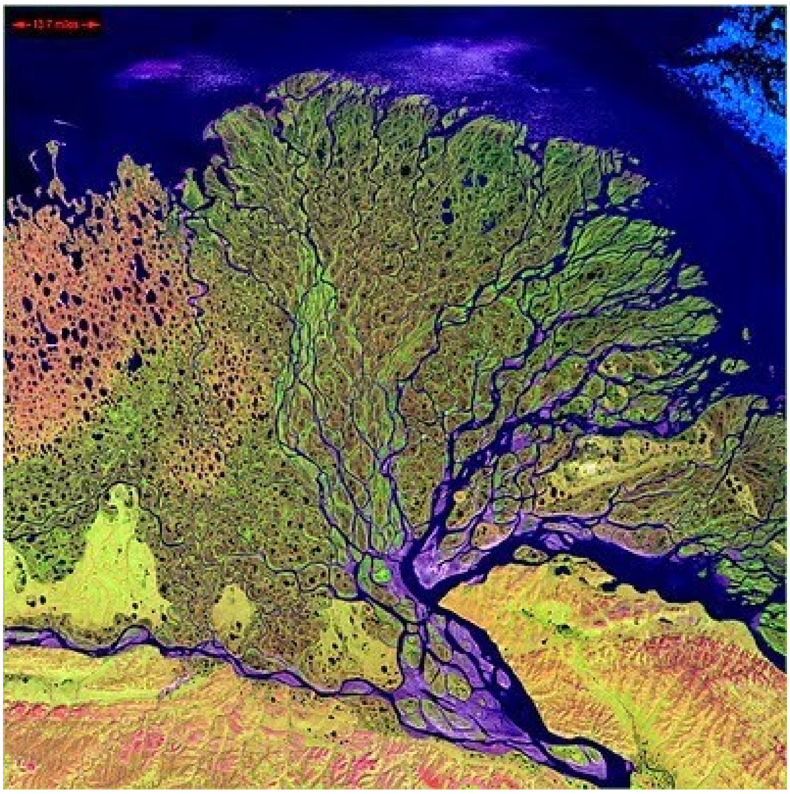


Figure 5.2: The Lena delta

Mandelbrot the similarity between the generated function with fractal dimension 1.2 and the skyline of a mountain. It was therefore obvious that Brownian motion should be extended by a second dimension in order to generate terrains.

At the beginning of the 20th century (1905), it was Albert Einstein who described this molecular movement mathematically. Finally, in 1923, Norbert Wiener succeeded in proving the probabilistic existence of this process modelled by Einstein. This criticism then called B. Mandelbrot and John W. van Ness onto the scene. With their groundbreaking work "Fractional Brownian motions, fractional noises and applications, SIAM Rev. 10 (1968), 422-437", the two succeeded in generalising the Wiener Pozess to fractal Brownian motion.

According to Einstein's explanations, the disorderly teeming of particles in liquids is due to the thermal motion of the liquid molecules that nudge the particles. With this explanation, Einstein helped lay the foundation for research into various group processes.

Today, scientists from a wide range of disciplines, from biology to sociology, use these findings. In very practical terms, it can be used to describe the foraging behaviour of ants, among other things: Ants usually follow the scent trails left by their colleagues". The paths of the insects follow these

Figure 5.3: The coast of England

chemical markers and show them the way home or to a food source. In order for them to find new food sources, a random deviation from these olfactory markers is important. These disturbances can be described by a random factor that appears as early as Einstein as the undirected influence of the thermal motion of molecules. This random factor, which the Frenchman Paul Langevin formulated more generally three years after Einstein, also serves in today's models as a variable for an undirected, seemingly random force that represents various chaotic influences.

In [**2**], many other terms related to Brownian motion are defined and explained. This gives us a concept that is of fundamental importance for both physics and the theory of fractals.

5.2 Fractals in the plant world

You can see in all plant structures that it is again a matter of accommodating a large surface area in a small space (Figs. 5.5 and 5.6).

Each slice of the mushroom colony is a two-dimensional fractal in itself, all slices together form a three-dimensional fractal.

Figure 5.4: Robert Brown (1773 - 1858)

Figure 5.5: Mushrooms on a tree stump

Figure 5.6: A well-developed cactus

5.3 Fractals in the animal world

Why are there actually both mice and elephants? Why can mosquitoes and dragonflies, sardines and tuna, hummingbirds and condors live in the same environment?

These questions are by no means as far-fetched as they sound. Rather, they arise from some obvious considerations. For one thing, the elementary components of both large and small animals, the body cells, are astonishingly similar. This is especially true when - as is the case with land mammals - the basic structure of the animals is the same. An elephant does not have larger liver cells than a mouse, but only many more. Measured against the differences in size between animals, the differences between their liver (lung, kidney) cells disappear. In particular, isolated cells in the test tube show about the same metabolic activity no matter what species they come from. The amount of heat produced by a given cell per unit time is approximately the same for mice and elephants. The same is true for oxygen consumption or, in the case of a kidney cell, fluid throughput. [**33]**

According to this, the metabolic heat production of an animal would have to be proportional to the number of its cells or, what amounts to the same thing, its body volume. Heat emission, on the other hand, is - apart from details such as fur, fat layer, sweat glands or plumage - essentially proportional to the surface area. However, both variables must be in equilibrium with each other, because otherwise the body temperature would inexorably rise or fall into unphysiological ranges.

If one animal is ten times as long, wide and high as another, it has a thousand times the volume but only a hundred times the surface area. Its ratio of volume to surface area is ten times as large, and thus also - all other things being equal - the ratio of heat generation to heat emission. Equilibrium could therefore only prevail at a certain optimal body size. An animal that is too large in this sense would have to make additional efforts for heat dissipation, an animal that is too small would have to constantly reheat in order not to cool down.

What has been illustrated here using the example of heat balance applies to numerous other metabolic activities such as respiration, digestion and urine excretion: Their intensity should actually be proportional to the volume of the associated organ, but they all take place on surfaces.

Such surfaces are, for example, lung alveoli, intestinal villi, interfaces between blood and cells or - for the exchange of substances between the complete blood circuits of mother and child - the placenta. Since surfaces with increasing body size grows more slowly than volumes, an elephant lung would have to perform far worse than a mouse lung.

However, we know that all animals living today are the results of a very long Darwinian evolution. In a selection process lasting millions of years, those animal species that make the best use of the available resources have always prevailed. An additional heating or cooling unit of any kind would require so much effort that the resulting disadvantage would far outweigh any advantages of a different body size. Accordingly, at least with the same body plan and environmental conditions, there should be very little variation in body size.

Obviously, there is something wrong with these considerations. Not only do large and small animals exist, contrary to this theory they also have their thermoregulation well under control; and this cannot be explained by fundamentally different construction plans. A mouse does not freeze to death as long as it has enough to eat, and hot-running elephants do not occur in nature either.

Only recently (1985) has a functionally based interpretation been found, with the help of two new concepts. On the one hand, the organism, as far as its metabolic activity is concerned, is regarded as a bioreactor under the same aspects as are decisive for industrial plants, and on the other hand, from a mathematical point of view, as a fractal structure.

First of all, the technical point of view. An industrial bioreactor typically consists of a large liquid vat to which the substances to be reacted (the reactants) are continuously fed; elsewhere, the reaction products are tapped off. The reactor contains either a catalyst (usually an enzyme) attached to particles of a porous gel or a porous membrane, or living cells suspended in the solution. In both cases, reactants and catalyst are not simply dissolved and thus distributed in the liquid in a freely movable manner; rather, the chemical reaction only takes place on the surface and inside the catalytically active part - gel or membrane or cells. This is called heterogeneous as opposed to homogeneous catalysis. In the long run, a flow equilibrium of the substances involved in the reaction is established in the reactor.

In order for the reactor to work efficiently, i.e. with a high mass transfer per volume, fresh reactants must be constantly fed to the active surfaces. There are two kinds of resistance to this: There is an outer boundary layer at the active surfaces, which is less mobile than the rest of the liquid; and the reactants have to enter the interior of the porous membrane by diffusion.

of the cell. To keep the second obstacle small, the thinnest possible membranes and small particles or cells are used; to mix up the boundary layer, the liquid is kept in intensive, turbulent motion with a stirrer (CSTR - continuously stirred tank reactor). The advantage of turbulent flow over laminar flow is the thorough mixing of the liquid; however, it comes at the price of a high energy input for stirring.

One should assume that the material turnover of such a reactor is proportional to its volume: In a vat twice as large, twice as many chemical reactions would have to take place per unit of time - all other things being equal. However, there are generally significantly less than twice as many.

As with allometry in biology, this empirically found dependence is described by a power function V^b with non-integer exponent b. The numerical value of b depends strongly on the cell or particle density in the reactor. At low cell densities and sufficiently turbulent mixing, the total turnover of the reactor is simply the sum of the turnovers of the individual cells in the suspension. In this case, the turnover is indeed proportional to the volume (b=1): The system behaves isometrically. At high cell densities, however, it is noticeable that the efficiency of the stirrer decreases with the size of the reactor, and the specific turnover decreases allometrically with the size of the system (b<1).

From the technician's perspective, a living organism is also a bioreactor with heterogeneous catalysis. The liquid, driven phase is, for example, the blood, the stationary phase, at whose boundary and within which the reactions take place, the body tissue. As in the technical reactor, a flow equilibrium of the chemical components is maintained through reactions and transport.

However, a decisive difference lies in the mixing process. In the organism there is no agitator, only the heart, which keeps the blood moving. This movement is almost without exception laminar and thus requires much less energy than the turbulent one. Nevertheless, there is extremely effective mixing between the liquid (blood) and the stationary phase (tissue). A substance injected intravenously into the human circulation is already equally distributed throughout the body after about two minutes. This result could only be achieved with a very high effort in a stirred technical reactor of the same volume.

The same arguments that apply to technical reactors also apply to the scale dependency of the turnover: since the reactants in an enlarged organism have to cover longer transport routes - and because the viscosity of the blood means that its flow velocity cannot increase to the same extent -, the higher the size of the organism, the lower the turnover.

body size the specific turnover rate. This would already be the beginning of an explanation for the allometry. However, natural reactors function several orders of magnitude better than technical ones: in a bioreactor with 109 cells per litre, the specific turnover rate falls with increasing volume according to an allometric exponent b-1 of about -0.3, whereas in organisms it is only -0.25, and that with the thousand-fold higher density of 1012 cells per litre!

It is therefore not primarily astonishing that the specific turnover drops with increasing body size, but that this drop is so small. Obviously, the mixing process in living organisms is many times more efficient than simple stirring. This in turn is due to the fractal structure of the vascular system and body tissue in general.

The metabolic intensity of an animal does not increase proportionally to its volume, i.e. to the third power of the body length. Nor does it increase proportionally to the surface area, which would correspond to the square of the body length or $V_{-}(2/3)$, where V stands for the volume. Rather, the rate of increase lies in between. A regularity of the form V^b applies, whereby the exponent b, determined from measurements on a large number of animal species, has approximately the value 0.74. Expressed in body length L rather than volume V, the metabolic rate scales as $L^{3b} = L^{2,22}$. In particular, the metabolic rate per unit volume decreases with increasing body size: this so-called specific metabolic rate is approximately proportional to V^{b-1} . [**34]**

The ratio of "surface area" to "volume" is therefore constant from the point of view of fractal analysis. This also means that it is no longer surprising that there are large and small animals. Mice and elephants are equally efficient in their metabolism and can therefore coexist without further ado. The paradox of allometric metabolic reduction mentioned at the beginning only comes about because we have related performance and functions to the wrong, because unstructured, volume or surface measure.

A fractal, like any mathematical concept, is not realised in nature with absolute perfection, but only approximately. In particular, self-similarity only applies within certain scale ranges with lower and upper limits. In our example, the upper limit is the size of the entire organ, the lower limit the size of a single cell. As the limits are approached, the fractal properties disappear or possibly change to a different scale behaviour.

An organism, considered as a whole, is an extremely inhomogeneous fractal. For example, the dimension D=2.22, as we measured it for the arteries, only applies to the transport-limiting part of the vascular system and dependent quantities such as the scaling of metabolic rates for the organism as a whole. In other areas, the capillaries of the vascular system itself or the alveoli of the lungs, the fractal dimension can reach much higher values, up to D=3, which means that these areas behave isometrically (i.e. scale with the volume).

In the animal world one finds many beautiful fractals. Under water, corals are certainly the most beautiful (Fig. 5.7).

Figure 5.7: Corals

Scientists studied male and female partridges (Alectoris rufa), both of which have intricate black and white plumage patterns on their chests.[**19]**
"We have shown that fractal geometry can reveal biologically significant information encoded in a complex plumage feature: the black spotted bib of the red-legged partridge," the researchers write in an article published 23 January in the Royal Society's journal Proceedings. "Our correlative results show that both better conditions and a better immune response can be predicted from bibs with higher FD."

The scientists also studied the relationship between bird health and fractals by restricting the diet of 33 male and female partridges during their moulting period, while 35 others could eat as much as they wanted. This resulted in the first set of birds weighing about 13 per cent less than the control group. The researchers then photographed the bibs of the partridges and found that the complexity of the fractals was significantly reduced in the birds whose diet was restricted. After losing weight, the same birds grew in plumage with a lower fractal dimension than before, while the birds whose weight remained constant showed no change in fractal dimension. Overall, the study showed that the fractals of birds tell us a lot about the health of individuals.

Figure 5.8: Camouflage is often included in the fractal structure

6 Fractals Art

After the many interesting and beautiful images, it is natural to look a little more into the relationship between fractal structures and various forms of art.

6.1 Fractal Music

The intellectual aspect of music is reflected, for example, in imitative techniques (kanon, fugue) and motivic work. This mathematical and logical aspect of music suggests the use of computers today [**10**]. One of the main areas of the author Arturo Raffaele Grolimund's intellectual preoccupation with music is the relationship of chaos theory or fractals to music. In his treatise
"Fractals of Music" from 1994, he examined fractal structures that occur in music itself. He shows very remarkably that all the important properties of fractals can also occur in music.

- The concept of self-similarity means the fact that a figure contains itself again and again on a smaller scale. This can be the starting point for creating musical structures.
- Here we also find the sensitive dependence on the initial conditions that is typical for fractals; how the canon or the drawn fractal develops depends essentially on the choice of the first tones.
- The piece Colours of Africa is inspired by African mbira music. The piece contains many fractal canons: with the same number of notes, twice as many canon voices can be accommodated. If you listen to the sounds, you can hear undulations.
- The prime example of a musical fractal is the overtone series, the physical basis of the major triad and thus of the traditional musical sense of harmony in general. It too is self-similar, i.e. it contains

itself infinitely many times in a regular way. These musical fractals or their graphics are not as complex as the Mandelbrot set or the Julia set. This is probably because each tone point already contains the overtone series as a timbre and is therefore itself a simple fractal.

- Likewise, with fractal polyphonic structures, the contrapuntal density can hardly be increased.

The old demand for the naturalness of art thus takes on an additional meaning. Nature and its processes serve as a model for innovative artistic design. The principle of accommodating a maximum of a certain substance in a small volume has often been mentioned. This is also evident here.

Apparently, fractal music was very popular in the early 90s. This is also shown in an article [**11**] by a group of composing programmers. A random algorithm varies the overtones of a sound containing overtones so that all harmonic proportions are avoided. This random variation in the micro-range merges into a similar variation in the macro-range, thus becoming audible and f i n a l l y expanding to such an extent that the entire 128-tone range of the general midi sound modules is seized.

- Choral: Harmony sequences are calculated algorithmically. The programmer has good training in four-part harmony, but it is demonstrably of no significance. The percussion and the bass provide a secondary effect with deliberately placed random structures.

- Lonely motif between major and minor: A lonely motif is morphed algorithmically. A programme that generates major and minor triads is an encouraging second, whereby key inputs keep the output within certain limits. A drum stabilises the action, which is unsettled by a rhythmically loosened whole-tone bass line.

- The somewhat friendlier live play: The algorithm used keeps lonely cells alive. The musicalisation of the live process is modelled on occidental score-reading habits.

- Developing variation: With his idea of developing variation, Arnold Schönberg anticipated the process of the recursive definition of fractal processes in a compositional way. The present programme takes up this idea and develops variations of well-known motifs according to the growth formula of Verhulst.

- Fractal Workstation '94: The programme was developed for the "Open Ears Day" of the music department in 1994. Based on the Höpfer algorithm, which was presented as a graphic miracle in "Spektrum der Wissenschaft" in 1986, amazingly different image formation processes are orchestrated.

- The rabbits on Wangerooge: The best-known fractal algorithm has the formula $X(n+1) = X(n)^2 + C$. When C is changed, it leads to surprising bifurcations that express themselves musically in motif ramifications. In this orchestral piece, eight Ataris play the same core programme in eight different rhythmic patterns (set by the conductor's Atari). The result is a farewell symphony to the hares of Wangeroog.

There are many more activities that can be found on the internet:

- Composer Dmitry Kormann from Brazil transcribed visual patterns into sounds using synthesizers and programs.

- Another important composer was the Spaniard Francisco Guerrero Marín.

- Phil Thompson and his `Organised Chaos`. He developed the Lebku- chen programme, with which fractal compositions are created.

The drum kit is possibly the instrument that is particularly suitable for fractals (fig. 6.1) Even the best drummer produces tiny fluctuations in rhythm and volume when playing. But these are by no means random. Instead, they form fractals: the deviations are self-similar on several levels. It is almost impossible to create such patterns intentionally, but unintentionally they even seem to have a decisive influence on our music.

A very interesting work can be found at [**4**]. There, the Lorenz attractor of Fig. 6.2 is transformed into music. The Lorenz attractor is the strange attractor of a system of three coupled, non-linear ordinary differential equations:

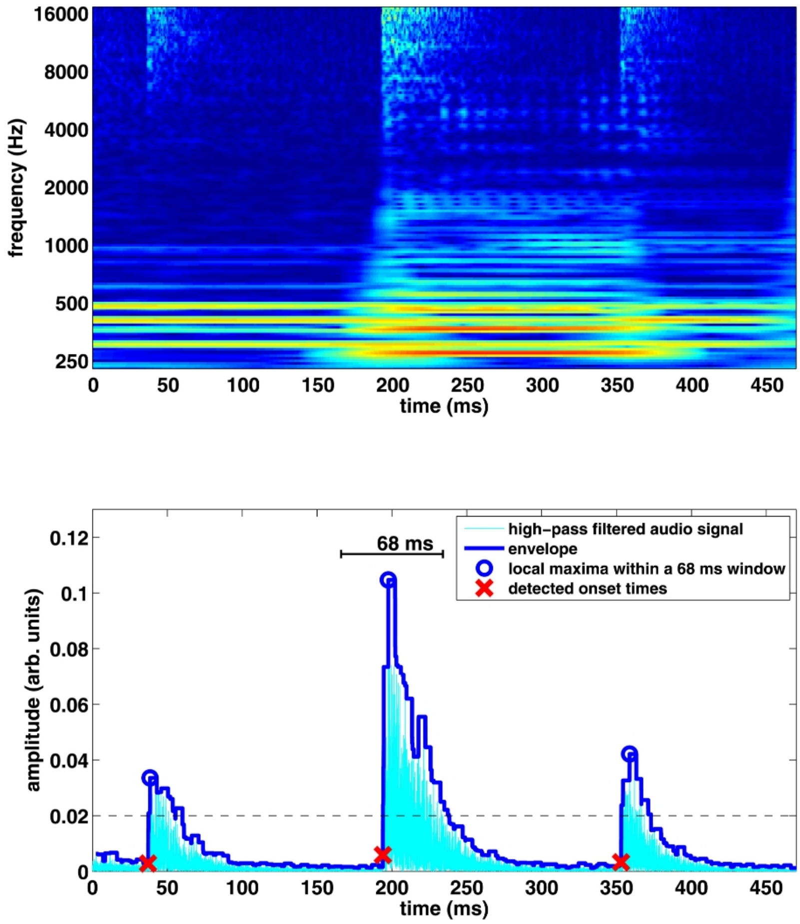


Figure 6.1: The accompaniment of a folk song by a drum

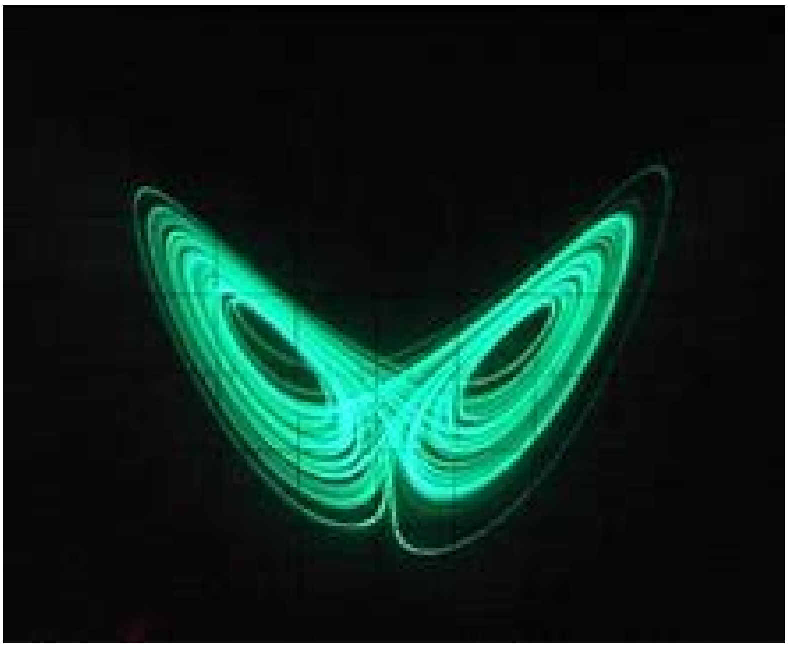


Figure 6.2: The Lorenz fractal

$$\frac{dx}{dt} = \sigma - (x + y) \tag{6.1}$$

$$\frac{dy}{dt} = R - x - y - x - y \tag{6.2}$$

$$\frac{dz}{dt} = -B - z + x - y. \tag{6.3}$$

Here B, R and σ are constants. At the beginning, one is at a certain point. Equations 6.1, 6.2 and 6.3 give a speed for each point (x, y, z), and after a certain time (one tenth of a second) you are at a new point where you behave in the same way. In this way, a three-dimensional curve $x(t), y(t), z(t)$ is created, the properties of which can be influenced by the constants B, σ and R.

The resulting lines come arbitrarily close to each other without touching. The Lorenz attractor has a fractal dimension of $D \approx 2,06$. The values of the Lorenz contractor calculated for (x, y, z) are translated in an elegant way into tones which, after a certain harmonisation and a restriction to major, result in a corresponding fractal tone sequence. It would certainly be interesting to translate other fractals into music as well.

This kind of music could also be called chaotic music. The numerical solution of the system shows deterministic chaotic behaviour for certain parameter values, the trajectories follow a strange attractor. Thus, the Lorenz attractor plays a role for mathematical chaos theory, because the equations probably represent one of the simplest systems with chaotic behaviour.

On youtube you can watch and listen to many videos with fractal music. Many pieces are very beautiful, but some are also extremely atonal, depending on which fractal is used. But the images of the developing fractal are always extraordinarily impressive.

A guitar string vibrates and describes half a sine wave (Fig. 6.3). If we create an artificial knot in the middle of the string by gently preventing it from vibrating at this point, it doubles the frequency and sounds an octave higher. We can also make the string vibrate three times as fast (another fifth higher) or four times, and so on.

The following questions are interesting:

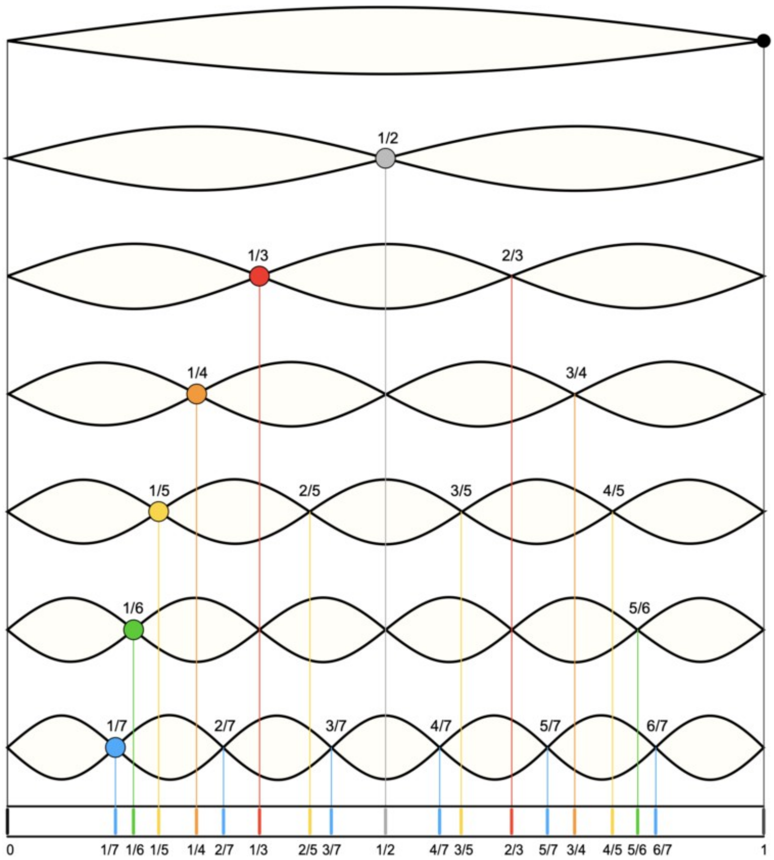


Figure 6.3: Vibrations of musical instruments

- Which harmonics are present?
- How loud are these harmonics in relation to each other?
- How do the volume and frequency of the individual harmonics change as the tone sounds?
- What background noises (touch noises, blowing noises, etc.) are added?

The following instruments have a particularly characteristic partial tone structure:

- String instruments have a very rich partial spectrum.
- Clarinets emphasise the volume of the odd partials.
- In the bassoon, the fundamental is much weaker than the first overtones.
- Bells often emphasise the thirds very strongly, and the overtone composition is complex.

- Tuning forks produce almost only the fundamental tone.

Every note is sound, and every sound is a fractal. And even in the midst of a whole orchestra, one recognises the oboe by its specific overtone series without ever having learned its structure. Its melody, too, is a succession of fractals.

For those who long for a stripped-down version of Bach - one in which three-quarters of the master's notes have been cut away - help is on the way. A scientist and his son, a musician, have developed a system for reducing music to its very essence.

In a paper published in the Proceedings of the National Academy of Sciences, Dr Kenneth J. Hsu, professor of geology at the Swiss Federal Institute of Technology in Zurich, and his son Andrew propose that mathematical extracts from the music of J. S. Bach could serve as matrices on the basis of which new compositions of Bach-like music could be created that are comparable in quality to those of the composer himself.

The system is based on fractal geometry, the study of structures where the shape of each small part reflects the shape of the whole structure. The Hsus believe that at least some pieces of music have a fractal form, as the basic musical patterns remain, even when notes are added or removed. J. S. Bach's Invention No. 1 in C major, for example, clearly has a fractal structure, the authors say. You can easily find this little piece on youtube.

In the introduction to their work, the authors recall that when the Austrian Emperor Joseph II first heard the opera The Abduction from the Seraglio, he told Mozart that the music was heavenly but contained too many notes. (Mozart is said to have replied indignantly that not a single note could be saved). Dr Hsu concedes that "music digests" suitable for the Austrian emperor and other philistines, consisting of "half-Mozarts, thirds-Chopins or quarter-Bachs" may never become popular. But the mathematical re- duction of compositions by the masters to a skeletal form could help composers create important new works, he believes.[**21**]

6.2 Fractals Images

Fractal painting is a relatively widespread sub-form of digital art and deals with the creation of digital images that essentially consist of one or more fractals (Fig. 6.4). As an independent art form, fractal art is controversial even among artists, as some argue that one is merely depicting parts of a mathematical-geometric form as a work of art, which does not reach the independent level of creation. There are various programmes that are used to create fractal art, including the freeware programmes `Apophysis` and `Apo7x`, `Fractal Explorer` and `Ultra-Fractal`.

Figure 6.4: Fractal artwork by Sebastian Baumer

Ornaments and ceramic products can also be derived from fractal structures (Fig. 6.4 , Fig. 6.5 and Fig. 6.6).

6.3 Fractals in the literature

Some time ago, physicists were also able to detect fractal structures in the great works of world literature. The researchers led by `Stanislaw Drózdz` from the Tech-

Figure 6.5: A fractal carpet

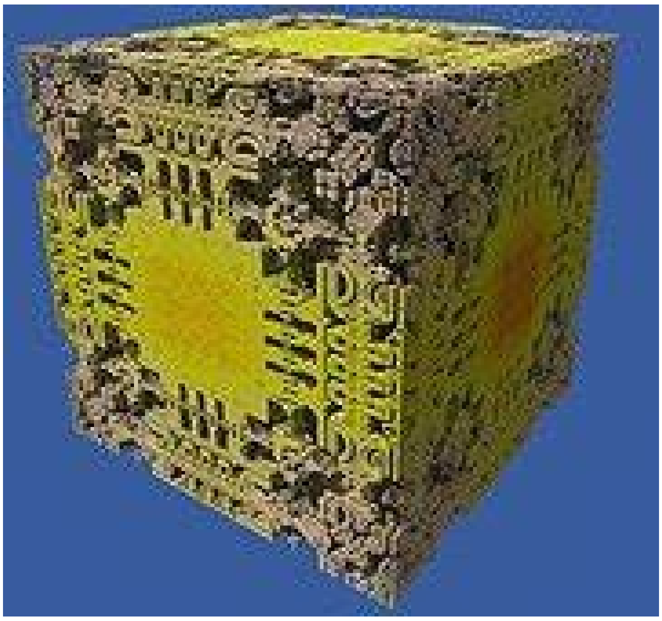


Figure 6.6: Fractal ceramics

niversity of Krakow analysed the number of words in sentences - in 113 books.

They fed their computers works by authors as diverse as Ja- mes Joyce, Thomas Mann, Honoré de Balzac, William Shakespeare, Virginia Woolf, Umberto Eco, Fyodor Dostoyevsky, Henryk Sienkiewicz, J. R. R. Tolkien and Julio Cortázar and looked for self-similar sentence length ratios in each case. "All the works tested show self-similarity in terms of the sequences and frequencies of their sentence lengths," the scientists report in the scientific journal
"Information Sciences": "These texts are all fractal." Most of the works turned out to be linear fractal. The researchers found by far the highest mathematical complexity in "Finnegans Wake" by James Joyce. In this difficult-to-understand work, the results of the analyses were indistinguishable from ideal, purely mathematically constructed multifractals. Finnegan's Wake is considered to be a
Joyce's work is one of the most remarkable, but also one of the most difficult to understand works of 20th century literature, which is partly due to its unusual language: Joyce coins his own language by reassembling, rebuilding, separating English words, or even mixing them with words from dozens of other languages (Portmanteaux). The result defies linear understanding and opens up possibilities for multiple interpretations.

New meanings are revealed to the reader with repeated readings. Roland McHugh has published concise annotations to Finnegans Wake in his standard work "Annotations to Finnegans Wake", in which explanations are given on the same page for many of the words used in the Wake, for example in the form of geographical references or references to languages from which the respective word or variants of it could have been borrowed.

But the Polish physicists found a remarkable exception: the Old Testament also stood out because of its particularly complex multifractal structures. It would certainly be an interesting question whether the level of a book can be expressed by its fractal dimension.

A multifractal system is a generalisation of a fractal system in which a single exponent (the fractal dimension) is not sufficient to describe its dynamics; instead, a continuous spectrum of exponents (the so-called singularity spectrum) is needed. The behaviour around each point is described by a local power law:

$$f(\vec{x} + \vec{a}) - f(\vec{x}) \sim a^{h(\vec{x})} . \qquad (6.4)$$

The exponent is called the singularity exponent because it is the local degree of the

singularity or regularity around the point.

7 Fractals in business and technology

Many processes in business and technology have a fractal character written all over them. Production figures for certain products, the demand for different raw materials, stock and exchange rates are characterised by a constant up and down.

7.1 The Fractal Factory

The idea of the fractal factory, as introduced into the discussion by Hans-Jürgen Warnecke in 1992, has found its equivalent in Industry 4.0 and the Smart Factory. The main characteristic of the fractal factory is its emphatically decentralised organisation in order to make the complexity in companies and their environment more manageable. Already in the preface to his book "The Fractal Factory. Revolution of Corporate Culture", Warnecke distinguished his approach from that of the "Computer Integrated Manufacturing (CIM)", whose deficits were becoming increasingly apparent at the time:

The underlying deterministic world view with known or, with the appropriate amount of research, recognisable connections between cause and effect is not sufficient, as it only applies to delimited sub-areas of reality. In contrast, Warnecke recommended accepting reality, i.e. acknowledging the non-linearity of processes in the economy and society, whose influence no company or factory can escape. There is no other way to meet the challenges of the future. The previous forms of service production have reached a very high degree of maturity. They are thus in a state where no greater benefit can be achieved, no matter how much effort is required. As a design principle to be able to bring the emerging new dynamics under a common denominator, Warnecke offered the concept of "service delivery", which was relatively new at the time.
"Fractal" on:

"Fractals communicate directly with corresponding fractals of suppliers or customers. Fractals can be distributed worldwide. Through self-organisation during

In each case, they select the methods for planning and control and use the automats and computers that are appropriate for fulfilling their tasks. A common feature of all industrial revolutions is the tendency from centralisation (steam engine or computer centre) to decentralisation (electric motor or workstation). Furthermore, an increasing worldwide availability and unstoppable spread can be observed in each case.

The modular structure offered itself as a suitable organisational form for the fractal factory. In terms of information systems, there was sympathy for the idea of the bionic production system, which originated in Japan and worked with neural networks and fuzzy logic, among other things - known today as Bionic Smart Factory 4.0.

A fractal factory is defined as an open system consisting of units that act independently and are self-similar in their goal orientation, forming a vital organism through dynamic organisational structures. The image of the "fractal factory" closely follows the mathematics of fractals for describing natural structures. Three essential properties of fractals emerge from the definition:

- Self-organisation,
- Self-similarity
- Dynamics.

The fractal thus presents itself as an independently acting corporate unit whose goals and performance can be clearly described. The self-similarity and self-organisation is reflected in the target system, which results from the fractals' goals, is free of contradictions and serves to achieve the corporate goals. Fractals practise operational, tactical and strategic self-organisation. The constant adaptation and optimisation of the decentralised organisational structures ensures a high degree of systematicity. For this purpose, the fractals are networked with each other via an efficient information and communication system. The supply of information follows the "fetch principle", the fractals determine (according to need) the type and scope of their access to the data. The boundaries between company divisions and also between companies are blurred, they enable the exchange of information and thus cooperative, partnership-based relationships with customers and suppliers. Between fractals there are workflow-functional, process-related connections that are established in the interest of the overall system. In principle, an attempt is made to manage processes holistically and not to build up unnecessary interfaces. In this way, the structure takes into account the aspects of speed and situational adaptability.

Invoice.

The structuring must be done in such a way that both the fractals and the entire factory are subjected to the same system because of the self-similarity. Experience from a number of projects shows that a company can be practicably described on the basis of a resolution of six levels. The advantage of this concept is that the levels can be considered and processed continuously, but also separately from each other.

- The cultural level describes the cultural elements, i.e. the values in the company. Guiding principles, common values and principles for dealing with each other and with the outside world are shaped. Furthermore, a place is assigned to one's own organisation and a purpose is given.

- At the strategic level, the way in which the resources available in the company are used to achieve a given goal is determined. A corporate target system must be developed.

- The socio-psychological level comprises the totality of all psychological, social and informal factors that determine the relationship structure of all employees of the company. Organisational structures and communication can be identified as the central factors of these levels.

- The economic-financial level of the fractals deals with the mode of charging for services. The focus here is on economic parameters with regard to economic efficiency and performance relevance.

- The informational level is primarily concerned with the design of the formal information flows and thus above all with the process organisation.

- In addition, it is important to distinguish between the process and material flow levels on which the arrangements and flow relationships in the company are designed with regard to the target system.

The level concept not only supports the representation of the entire company, but also depicts the individual fractals. It thus serves to reduce the complexity of companies by creating structures while maintaining the holistic nature of task fulfilment.

You can see in Fig. 7.1 that the same structure is repeated on all levels. Official communication only takes place via the tips of the triangles. Thus a

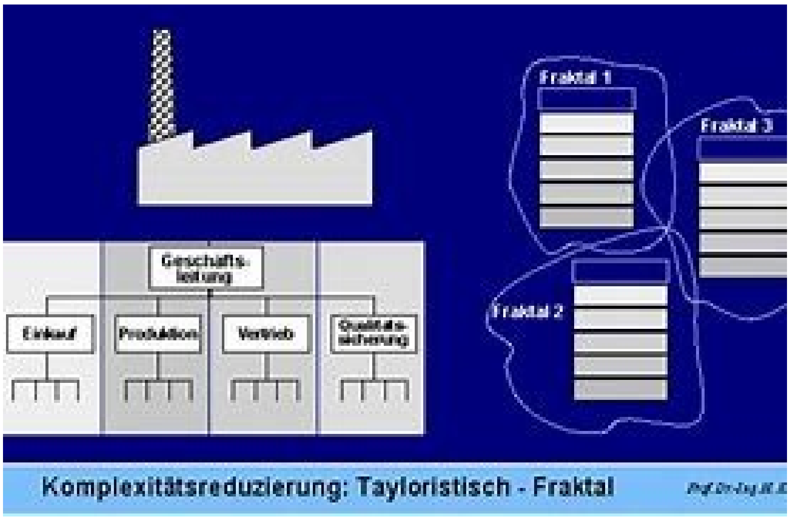


Figure 7.1: Structure of a fractal factory

The result is a harmonious structure through which disturbing influences can be compensated for very quickly.

7.2 Fractals in the financial world

All time-ordered data in the financial world show very pronounced fractal structures.

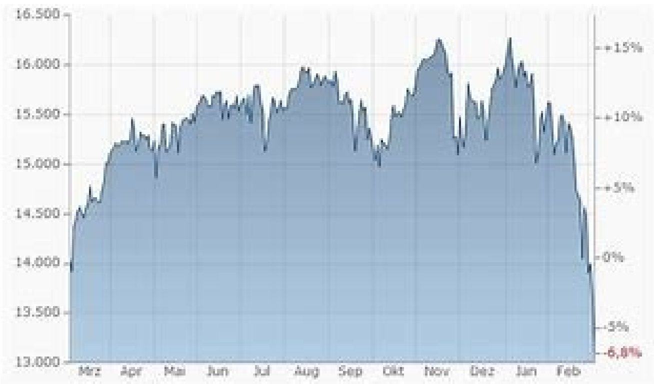


Figure 7.2: The DAX - price over the course of a year

The curves always look the same, whether you look at them over a day, a week or a month.

There is a simple explanation for this. You have a certain amount of securities that could be sold on the stock exchange, which would generate a certain amount of revenue. If the price falls, these proceeds become smaller, so one would be

Figure 7.3: The DAX - price over the course of 10 years

Figure 7.4: The DAX - Price since 1988

do not sell. If the price rises, the proceeds become larger. Now it depends on the investor's discretion what profit value he is satisfied with. At a certain point of the rise, he (and possibly others) decides to sell; he has realised a certain profit. If several investors follow this course of action, the price falls, and this is how the ups and downs in the curves come about. Here many investors act more or less speculatively; they buy a share from which they expect such a rise and sell it again if the rise comes about and the profit meets their demands.

It is interesting to compare the curves of New York, Tokyo and Frankfurt. They are almost identical, only shifted in time. The time difference between Tokyo and Frankfurt is seven hours, the time difference between Frankfurt and New York is six hours and the time difference between New York and Tokyo is even eleven hours. These time differences are exploited and lead to so-called seconds trading. For example, you buy securities in Tokyo at a certain price, the price rises, and the Frankfurt stock exchange starts with the higher price; you can take advantage of this rise and immediately realise a profit in Frankfurt.

Trading in seconds is a particularly fast form of day trading in which value shares are bought and sold again within a very short time. While day traders usually trade shares within a day, in seconds trading the shares are often held for only a few seconds to a few minutes. The fast transactions were made possible by the computer exchanges with their prompt execution, but only since real-time trading via the internet have normal private investors also been able to engage in day trading or seconds trading. A particularly important rule in seconds trading prohibits holding a security overnight so as not to suffer losses during this time. In order to be successful in seconds trading, it is extremely important to be able to react immediately to events that influence the market.

A frequently used form of representation are candlestick charts. They are used in the analysis of stock market prices to depict the price trend of a share or another exchange-traded security.

Each individual candle shows how the price has developed within a certain period of time. In a daily chart, each candle represents the price movement within a specific day. If, on the other hand, a weekly chart is viewed, each candle shows the price movement within a week. The individual candles show the first price (opening price), the last price (closing price), the highest price and the lowest price of the period under consideration.

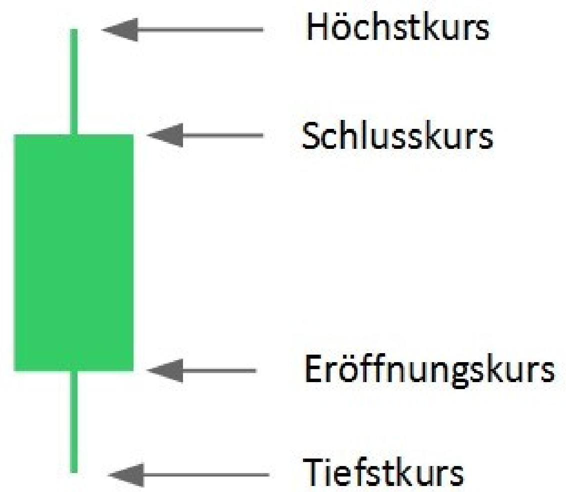


Figure 7.5: The candlestick chart for one day

Each individual candle consists of a box-shaped body in the middle of the candle and two lines protruding above and below the box. The upper and lower limits of the body indicate the respective opening price and closing price. If the closing price is above the opening price, the box turns green.

drawn in. In this case, the price has risen in the course of trading. If, on the other hand, the price has fallen, the closing price is below the opening price and the box is drawn in red. Occasionally, the price closes at the same point at which it opened. In this case, a straight line is drawn in place of the body. This candlestick shape is called a doji. The two lines above and below the body are called shadows. The top of the upper shadow indicates the highest price of the period under consideration, while the lower shadow marks the lowest point.

The shape of the candles tells us a lot about how the price has developed in the period under consideration. From the length of the shadows and the size of the bodies it can be concluded how the price has behaved in the period under consideration. Some examples of this can be found in the charts shown (Fig. 7.6 and 7.7).

Figure 7.6: The representation of an upward movement

1 A long body with small shadows indicates a strong upward movement (green body) or downward movement (red body).

2 If, on the other hand, a candle has only a very small body and at the same time long lower and upper shadows, this indicates a strongly fluctuating trade without a clear trend.

3 A very long shadow in one direction indicates a trend reversal. A long lower shadow indicates that the price fell sharply at a certain point. However, a strong countermovement then began and the price was able to recover most of its losses by the end of the trading session (Fig. 7.6).

4 Exactly the opposite is the case with a long upper shadow. Here the price rose strongly, but could not hold its high and in the end had to give up most of its interim gains.

5 If the highest price and the lowest price of a candle are very close to each other, this indicates uneventful trading.

6 Gaps are gaps between two candles. These price gaps occur when the following candle opens clearly above (upward gap) or below (downward gap) the previous candle. A gap only remains visible on the chart if the price does not move back in the direction of the previous candle and thus closes the gap.

Figure 7.7: The representation of a downward movement

An upward gap indicates that there were significantly more buyers than sellers at the opening of trading. Conversely, a downward gap indicates that there were significantly more sell orders than buy orders before trading opened Fig. 7.7. Upward trends, downward trends and sideways movements In a candlestick chart, upward trends, downward trends and sideways movements are relatively easy to identify.

In an upward movement, the candles close above the closing price of their leading candles. Within the upward movement, there are clearly more green candles than red candles. In a downward movement, on the other hand, the red candles predominate. Most candles within the movement close below the closing price of their predecessors. As the name suggests, the price moves sideways in a sideways movement. The candles move up and down within a narrowly defined range. In the sideways movement, the candles often change colour.

Next, we look at how the individual candles in the candle chart relate to each other.

Figure 7.8: A compression zone

A frequently occurring pattern in the candle diagram are compression zones in which several consecutive candles oscillate up and down in a narrow box (point 1). The bodies of the individual candles all lie more or less on one line. In many cases, all the candle bodies also lie within the two shadows of the largest candle. Within this box, the price fluctuates back and forth between two narrow frames without being able to form a clear trend. If the price finally succeeds in breaking out of this box, there is often a violent, steep price movement in the direction of the breakout (point 2). However, before breaking out of the box, it is not possible to predict in which direction the breakout will take place. Therefore, it is advisable to wait until the first candle closes outside the box before entering (Fig. 7.8).

Significant highs or lows often form resistance zones for subsequent price movements. At these points, the probability is therefore higher that the price will bounce off the same point again and turn back in the opposite direction. These resistance zones are even stronger if the price has already bounced off the same point several times.

In Fig. 7.9 you can see such a resistance zone. The price has bounced off the same line twice, at point 1 and at point 2. This high is interesting for two reasons: firstly, the probability is particularly high that the price could turn downwards again at this point. At the same time

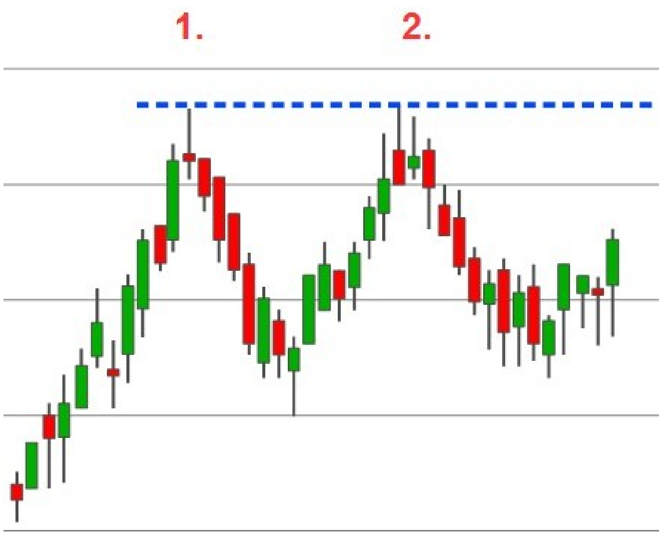


Figure 7.9: Resistance zones

but a break through this resistance line would be seen as a particularly positive sign. Many traders and investors see the break of such a resistance line as an indicator of further rising prices and therefore enter the value shortly after a breakout. Therefore, after a breach of this line, there is increased demand, which drives the price further upwards. Before entering, however, it is advisable to wait whether the candle closes at least above the resistance line. Sometimes it is precisely at such resistance lines that bull traps occur, where investors enter in expectation of rising prices, only to be confronted with a sudden countermovement shortly afterwards. As soon as the price has moved above the resistance line for a longer period of time, the resistance line becomes a support line. A support zone thus forms along the former resistance line, making it more difficult for the price to fall below the former resistance line again.

Candlestick formations are certain combinations of successive candles that give the investor indications of future price developments.

While compression zones, support lines and resistances can also be seen in other charts, such as line charts or bar charts, candlestick formations can only be discovered in the candlestick chart.

An example of a candlestick formation can be seen in Fig. 7.10. The last three candles in the candlestick chart form a so-called Morning Star formation. The first candle of the formation is a long red candle. Next is a small candle that opens with a gap. The last candle is a long green candle that also opened with a gap. The Morning Star candlestick formation predicts rising prices ahead. Therefore, rising prices are more likely in the following days. In general, candlestick formations can be divided into two groups:

- Reversal signals indicate that the price will turn in the opposite direction after the candlestick formation appears.
- Continuation formations, on the other hand, predict that the existing trend will continue.

Figure 7.10: A Morning Star formation

The Evening Star formation consists of three consecutive candles.

1 The Evening Star always follows an upward movement.

2 The first candle of the Evening Star formation is a long green up candle. If the chart under consideration is printed in black and white, the first candle would be a white-bodied candle.

3 The second candle opens with a price gap. The candle opens above the body of the previous candle. It is not necessary for the candle to open above the shadow of the previous candle. Just like the opening price, the closing price of the candle is also above the body of the first candle, so that the price gap remains visible on the chart.

4 The second candle is always a relatively small candle. In the Evening Star formation, the colour of this candle is unimportant. If the second candle is

candle around a doji (i.e. a candle where the opening price and the closing price are identical), the formation is called an evening doji star. The second candle opens with a jump in price, but then cannot maintain this initial momentum and ends up closing at more or less the same point at which it opened.

5 The last candle is a red downward candle. In the classic Evening Star formation, this candlestick must begin with a downward price gap. Many authors, including Steve Nison, who made candlestick charts popular in the West, attach little importance to this price gap and recommend trading the formation even if there is no gap. It is important, however, that this candlestick is a long distinct candlestick that clearly protrudes into the candlestick body of the first candlestick.

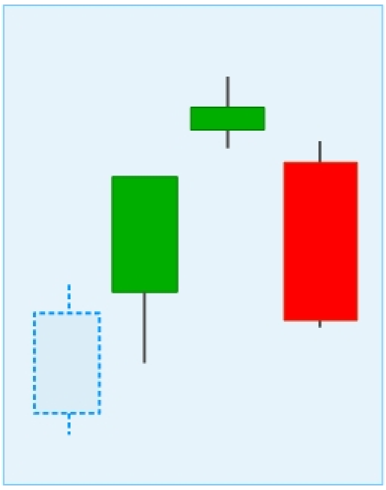


Figure 7.11: An Evening Star Formation

We also find fractional structures in exchange rates between different currencies.

- Weakly fluctuating shares beat the market.

  One of the core assumptions of this model is that shares can be classified according to their sensitivity to a market return beta. The higher the respective beta, i.e. the sensitivity to market returns, the higher the risk of the individual share. However, work from the 1970s shows that this is precisely not the case for certain shares.

- "Low Volatility" as a factor.

  In a systematic factor process, the volatility (e.g. over the past

Figure 7.12: The exchange rate for Euro - Dollar

three years) is calculated as a measure of the fluctuation for the shares of a market. This measure is also understood as a measure of risk. With this, the shares can be assessed with regard to their supposed risk in a regular sorting process, e.g. every six months. If one selects the top 30 percent of the shares for a portfolio, i.e. those that fluctuate the least, and holds them until the next switching point, a "low volatility" factor is created.

- Invest in undervalued and small stocks.

  A decade later, empirical capital market research focused on so-called "value" approaches, i.e. structured factor investments in undervalued shares. Classical ratios are used here, such as the price-to-book ratio, the price-to-earnings ratio or the dividend yield. These studies are based on the realisation that shares that are considered favourable (e.g. measured by the price-to-book ratio) perform better than shares that are considered "expensive". Further studies also showed that structured investment in smaller companies allows for excess returns.

  Both factors are considered risk factors here, since shares that have a favourable price-to-book ratio are often companies that are under economic stress and are traded at a discount. The same applies to small companies, which may have special balance sheet risks, for example. All in all, this follows the statement of the original model: the more risks the investor takes, the higher his expected return.

- Shares that have risen sharply: Momentum.

Finally, at the beginning of the 1990s, Jegadeesh and Titmann publish one of the most serious anomalies from the perspective of capital market theory. The author duo shows that stocks that have risen in the past (e.g. measured by the performance of the last six or twelve months) and are subjected to a systematic ranking perform better than stocks that have fallen recently. Stocks that gain momentum thus remain buoyant and have momentum.

- Are these temporary anomalies or is it structural?

  From the point of view of classical theory, excess returns to the market can only exist either through higher risks or otherwise only temporarily, as the markets moved into a process of equilibrium. Worse, the "low volatility" and momentum anomalies, in particular, exploit purely the price history and are thus a strong example of how even the weak form of the Efficient Markets Theory, which excludes just that, may not describe market phenomena precisely enough. It may now be objected that the examples above represent only a short period of time. However, some of these factors have been shown to exist well into the Victorian era (in the case of Momentum, as far back as 1801).

- So is it coincidence or is it structural? Will these anomalies disappear or is it the cause that investors do not act completely rationally after all and are subject to certain "biases", as behavioural economics assumes? And what about the other anomalies that have been found in the meantime, the so-called "factor zoo" and other discussions that also ask why momentum and "low volatility" are not actually included in the capital market models?

In one of his last works, B. Mandelbrot outlines his vision of trend-based, fractal markets. This is a sketch that uses signal theory methods to allow an alternative view of factors and market returns.

7.3 Fractal programming - its usefulness for education

Here I would like to cite the "Fractal Foundation" as an example that should certainly be emulated in many other places. Their internet presence begins with the

Statement: "We use the beauty of fractals to inspire interest in science, mathematics and art. Our vision is a world of wonder and curiosity, a culture of scientific inquiry, an appreciation for the interconnectedness of natural systems and an understanding of their essential non-linearity. We see a world where everyone is mathematically literate and understands how mathematics is a powerful tool to turn their visions into reality. We are out to change the culture. We have enough sports heroes and movie stars. What we need are science heroes and maths stars. Through large-scale public art, we promote the beauty of maths and science and showcase the creativity and intelligence of our students. MariaCThe Fractal Foundation was founded in 2003 and grew out of the Albuquerque-based Chaos Club. Since then, we have taught fractals to over 69,000 children and 55,000 adults. Based in Albuquerque, most of our activities to date have taken place in New Mexico, but they continue to expand."

First we recommend you download the amazing FREE fractal explorer XaoS

`http://fractalfoundation.org/resources/fractal-software/` ,

with which you can zoom into mathematical fractals.

Next, you can visit the page Fractivities

`https://fractalfoundation.org/resources/fractivities/`

where many different projects for home, school or nature will be presented. We would love to have you participate in the Fractal Trianglethon! We need volunteers, teachers and students to help build the largest Sierpinski Triangle in the world.

After that, you might want to try our online Fractal Course

`http://fractalfoundation.org/resources/lessons/`

which deals in detail with fractals in nature and mathematics. While it is aimed at high school students, much of the material is also suitable for younger and more advanced students.

There are also a large number of programmes offered that make it easy for students to get started in the world of fractals.

If you already have knowledge of programming, you can also find ready-made programmes in several programming languages.

On the website

`https://technik.blogbasis.net/ein-fraktal-programmieren-06-08-2013`

for example, a functioning Java programme is given that was already written in 2013.

8 Chaos theory

- Chaos and order

  According to classical Greek mythology, chaos was the disordered original state of the world. According to this, it was only from chaos that the later ordered world, the cosmos, emerged. This myth of an initially disordered primordial substance is also found in numerous other myths about the origin of the world. With the Christian doctrine of the creation of world order out of nothing, however, chaos lost its original meaning in late antiquity. Today, chaos is therefore generally understood to mean states and processes of unpredictability and unpredictability. Beyond that, however, chaos still has the reputation of something indescribable, uncanny or even dangerous. In the scientific field, people in most fields of research are alarmed when they encounter chaotic phenomena (such as turbulence, vibrations), and they often try to dismiss and exclude these observations as insignificant disturbances. In the social sphere, too, most people are afraid of chaos. It is not for nothing that extremists who want to destroy an existing political order through violent actions are often called chaotic. It seems to be a basic human need to bring the indescribable and uncanny of the world into a reliable order. In earlier times, people tended to cling to superstition rather than trust in scientific research. In Europe, the prevailing (church) doctrine until the 16th century was that the earth was a disc and formed the centre of the universe. The basis for this was the early work of the Greek geographer and astronomer Claudius Ptolemy (around 100-160), who had compiled an extensive series of maps of the then known world. Ptolemy also adopted the even older conception of the Greek philosopher and natural scientist Aristotle (384 - 193).
  - 322 BC), according to which the sun and all other heavenly bodies move around the earth (geocentric or Ptolemaic world system).

  The closed world view that emerged on this basis was the great treasure of the ancient sciences from antiquity to the late Middle Ages. This understanding of the world could rely on a centuries-old, traditional order.

which was also supported by the prevailing church doctrine and biblical interpretation.

- Determinism and reductionism

But this old order could no longer be reconciled with the actual observations in nature. The modern natural sciences that emerged in Europe in the 15th century therefore attempted to give the phenomena of nature a new order that could be verified by observation. A first important step in this direction was the work of the German-Polish astronomer Nicolaus Copernicus (1473 - 1543). Another important pioneer of the Copernican world system was the Italian physicist and mathematician Galileo Galilei (1564 - 1642), who not only discovered the laws of pendulum and fall, but also built an improved telescope and used it to discover the phases of Venus and four moons of Jupiter.

However, the research of the English physicist and mathematician Isaac Newton (1643 - 1727), who is still regarded as the pioneer of the modern natural sciences, proved to be decisive. Newton discovered the laws of gravity, the principle of recoil and the law of inertia; he discovered the laws of the light spectrum and developed a convincing theory of light and colours; and he developed the differential and integral calculus at the same time as the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646 - 1716). The work of the French astronomer and mathematician Pierre Simon de Laplace (1749 - 1827), who, among other things, precisely depicted the movements of the heavenly bodies, went in the same direction. He and his contemporaries saw the world as a clockwork whose processes could be studied individually, reduced to certain laws and subsequently generalised (reductionism). However, already at the end of the 19th century, the French mathematician Jules Henri Poincaré (1854 - 1912) came across a physical problem that called Newtonian determinism and Laplacean reductionism into question again. Poincaré, who did important work in the field of algebra (theory of automorphic functions, theory of physics), also investigated the so-called three-body problem. With the help of the laws of gravitation discovered by Newton, the movements of two celestial bodies can be calculated unambiguously (for example, the elliptical movement of the moon around the earth). However, as soon as a third gravitational body is added (for example, the sun with its gravitational pull on the earth and moon), the equations necessary for the calculation can no longer be solved unambiguously. Although the orbits can be approximately calculated in advance, a deviation soon occurs, so that long-term predictions are not possible. It also follows that it is impossible to say whether our solar system is actually stable - in any case, it does not behave like clockwork. However, this scientific insight of Poincaré was initially dismissed as a curiosity even among experts.[**39]**

- Since the 1960s, however, a new research direction has emerged that not only dismissed chaos as a puzzling special case, but was also interested in its properties. Researchers from various disciplines discovered that chaotic systems can organise themselves within the framework of a dynamic formation of order and can form astonishing patterns of order. These exceptions merely confirm the general rulelessness of chaos, but at t h e same time chaos can no longer be equated with randomness. The results obtained in the research of chaotic states and processes are summarised under the term "chaos theory", and today it influences numerous fields of natural science and the humanities. Many advocates of the new research direction are of the opinion that chaos research (along with quantum mechanics and the theory of relativity) is the third significant scientific achievement of the 20th century.

8.1 Nonlinear Systems

From a scientific point of view, chaos theory belongs to the research area of non-linear dynamics. Although there is no linearity in chaos according to cause and effect (causal relationship) and chaotic systems behave unpredictably and unpredictably, they naturally follow the laws of nature and are therefore not random. This is why chaos research also speaks of lawful (deterministic) chaos.

- Bifurcation,
- Phase space,
- Attractor,
- sensitive dependence,
- Non-linearity,
- exponential error growth,

- multiplicative self-similarity,
- Self-organisation,
- Baker transformation,
- Mixture, strong and weak causality.

The differential or difference equations that arise contain non-linear functions. Under certain circumstances, these non-linear equations show interesting features and solutions, for example surfaces in phase space as attrac- tors, self-similarity and fractal structures. Important applications of non-linear dynamics can be found, for example, in mechanics and astrophysics.

$$\ddot{y} + 2\frac{\dot{L}}{L}\dot{\Phi} + \frac{g}{L}\sin\phi = 0 \qquad (8.1)$$

$$L(t) = L_0 - \Delta L_0 \cos(\Omega t). \qquad (8.2)$$

These two equations describe the movements of a swing, for example. Here you find the self-similarity in a simple way. The curves are more or less the same, only the height and speed of the movements can differ. Since the sine function is applied to the zero derivative, it is a non-linear system. In the concrete case, the sine function limits the instability of the occurring parameter-excited oscillation, since the system is detuned into stable ranges at larger amplitudes. The non-linear component is the reason why the natural frequency depends on the amplitude of the oscillation.

The phase space describes the set of all possible states of a dynamic system. A state is uniquely represented by a point in the phase space. In mechanics, it consists of generalised coordinates (configuration space) and associated generalised velocities.

With n degrees of freedom, the phase space is 2n-dimensional. For example, a gas particle in three-dimensional space has n=3 degrees of freedom, with the associated impulses this is 6 phase space coordinates. A system (gas) of N particles has a 6N-dimensional phase space. However, phase spaces are also investigated in other applications outside mechanics.

The temporal development of a point in phase space is described by differential equations and represented by trajectories (trajectory curves, orbit) in phase space. These are described by first-order differential equations in time and are uniquely determined by a starting point (if the differential equation is time-independent, these are autonomous differential equations). Accordingly, two trajectories do not cross in phase space, since the further course is not clear at a crossing point. Closed curves describe oscillating (periodic) systems.

For systems with up to three variables, the phase space can be represented graphically. Especially for two variables, the movement (trajectories, phase space flow as a vector field) can be graphically represented and qualitatively analysed in a phase space portrait or phase portrait.

The historical origin of the use of phase spaces is often traced back to Jo- seph Liouville - because of Liouville's theorem (1838) that in con- servative systems (with conservation of energy) the phase space volume of neighbouring trajectories is constant in time. Liouville, however, did not have a mechanical system in mind, but proved the theorem for general ordinary differential equations of the first order; the connection to mechanics was first suggested by Carl Gustav Jacobi. The phase space concept only emerged after mathematicians switched to considering higher-dimensional spaces in the further course of the 19th century. The first use of phase space in today's sense was by Ludwig Boltzmann in 1872 as part of his investigations into statistical mechanics, which was adopted by James Clerk Maxwell in 1879. The concept was then used in Boltzmann's and Josiah Willard Gibbs' lectures on statistical mechanics, in the 1911 article on statistical mechanics in the Encyclopaedia of Mathematical Sciences by Paul Ehrenfest and Tatjana Ehrenfest (who introduced the term Γ for phase space) and in the qualitative theory of differential equations by Henri Poincaré.

A dynamic system whose trajectories fill the entire phase space, i.e. come arbitrarily close to every point in the phase space, is called ergodic, see also ergodic hypothesis. In conservative mechanical systems (closed systems), according to Liouville's theorem, the phase space volume of neighbouring trajectories is constant in time; in dissipative systems it decreases (open systems).

In Hamiltonian mechanics, the phase space is an example of a symplectic geometry, and Hamiltonian mechanics is the geometry of the phase space. Since the momentums are defined as derivatives of the Hamiltonian function according to the generalised co-ordinates, the phase space there is a cotangent bundle over the configuration space.

In quantum mechanics, the Heisenberg uncertainty principle expresses a quantisation of phase space. The Heisenberg Uncertainty Principle or Uncertainty Relation (more rarely also Uncertainty Principle) is the statement in quantum physics that two complementary properties of a particle cannot be determined simultaneously with arbitrary precision. The best-known example of a pair of such properties is location and momentum.

The uncertainty principle is not the result of technically remediable inadequacies of a corresponding measuring instrument, but is of a principled nature. It was formulated in 1927 by Werner Heisenberg within the framework of quantum mechanics. The Heisenberg uncertainty principle can be regarded as an expression of the wave character of matter. It is considered the basis of the Copenhagen interpretation of quantum mechanics.

The phase space portrait provides a possibility to graphically analyse the temporal developments of dynamic systems. For this purpose, only the dynamic equations of the system are required, an explicit representation of the time evolution, for example by analytically solving a differential equation, is not necessary.

As an example, here are some elements of phase space analysis in a two-dimensional system given by the differential equations

$$x' = \frac{dx}{dt} \quad , \quad y' = \frac{dy}{dt} \tag{8.3}$$

$$x' = f(x, y) \quad , \quad y' = g(x, y) \tag{8.4}$$

is described:

- Drawing in the vector field of the dynamics: For a grid of points, the direction of the movement in phase space is represented by arrows. If one now follows the arrow starting from a certain starting point, one comes to a new point where one can repeat this procedure. In this way, the vector field can also be used to draw typical trajectories in the phase space portrait, which help to assess the qualitative behaviour of the temporal development. In the van der Pol oscillator, for example, all trajectories run towards a limit cycle, which can be illustrated by means of example trajectories inside and outside the cycle. For simple dynamic systems, the vector field and example trajectories can often be drawn in by hand; for more complex systems, this can be done by computer programs.

- Drawing in the zero lines: A zero line is a curve in phase space along which one of the dynamic variables does not change. In the case of the above two-dimensional system, the x-zero cline is defined by the condition x'=f(x,y)=0 and the y-zero line is defined by y'=g(x,y)=0. These equations can often be solved for one of the variables even if the overall dynamics cannot be integrated analytically.

- Determining fixed points and their stability: Fixed points are states that do not change with time. Such fixed points correspond to the intersection points of the zero clines in phase space. In the above two-dimensional system, this is explained by the fact that at such a crossing point the condition $g(x, y) = f(x, y) = 0$ is fulfilled. A linear stability analysis can also determine whether trajectories near these points are attracted or repelled.

- Finding separation matrices: A separation matrix is a curve or surface that separates phase space areas with different behaviour from each other. For example, if there are two fixed points that attract trajectories, there may be a separatrix that separates the two catchment areas. With the locations and the stability of all fixed points or with the vector field of the dy- namics, the separatrices can be found without further calculations in suitable cases.

The sensitive dependence on the initial values is a central characteristic of chaotic dynamic systems. This is understood to mean the property of such systems to produce a completely different system behaviour over time with only an infinitesimally small change in the initial conditions. In this sense, mathematics speaks of deterministic chaos: the development of a chaotic dynamic system is unpredictable as a result of the unavoidability of measurement errors in the determination of the initial state, not due to stochastic behaviour.

Self-organisation is a form of system development in which form-giving or formative influences come from the elements of the system itself. In processes of self-organisation, structural orders or pattern formation are achieved without these demonstrably arising from external (externally organised), controlling influences or being able to be linearly assigned to specific causes. Self-organisation is a property of complex, dynamic systems, which are investigated in synergetics - the theory of the interaction of elements. This often spontaneous emergence of patterns of order from the system dynamics is referred to as emergence or emergent phenomena.

In political or organisational theory, self-organisation refers to the shaping of living conditions according to flexible, self-determined associations.

and is similar to the concept of autonomy. The political or organisational use of the word self-organisation is often wrongly legitimised with system-theoretical and scientific justifications, but is not directly related to these explanatory models.

In the case of self-organisation, a distinction can be made between autogenous (from one's own forces) and autonomous (self-determined) self-organisation:

- Self-reference: Self-organising systems are self-referential and exhibit operational coherence. This means that "every behaviour of the system has an effect on itself and becomes the starting point for further behaviour", i.e. it has a circular effect. Operationally closed systems do not act on the basis of external environmental influences, but according to the forms of information generation and processing that have developed within them, "quasi from within themselves". The results of internal processing change the initial conditions for subsequent processes. There is a self-reference and thus an informational unity.

- Path dependency: A development path that has been taken cannot easily be abandoned.

- Indeterminacy: The course of development is ultimately unpredictable. Indeterminacy depends on coincidences; a small change in the initial conditions can lead to completely different paths.

- Autonomy: Self-organising systems are autonomous if the relationships and interactions that define the system as a unit are determined only by the system itself. Autonomy refers only to certain criteria, since a material and energetic exchange relationship with the environment still exists. In vertical autonomy, decision-making freedoms of subordinate units are sharply separated. With horizontal autonomy, areas on one level are independent of each other.

- Centralisation and decentralisation: The delegation of decision-making powers at lower levels is referred to as decentralisation, whereas the delegation of decision-making powers at higher levels is referred to as centralisation.

- Redundancy: In self-organising systems, there is no separation in principle between organising, shaping or guiding parts. All parts of the system are potential designers. Several parts can organise the do the same thing, which provides a kind of abundance (= redundancy). Redundancy can increase autonomy because there is no strict division of labour.

In order to be able to speak of self-organisation, the following (interdependent) criteria must be fulfilled:

- the evolution of a system into a spatially/temporally organised structure without external intervention,
- the autonomous movement into smaller and smaller regions of the phase space (so-called attractors),
- the development of correlations or spatio-temporal patterns between previously independent variables whose development is only under the influence of local rules.

The behaviour of a self-organising system often shows very good properties in terms of scalability and robustness to perturbations or parameter changes, which is why self-organising systems are well suited as a paradigm for future complex technical systems. However, there is no simple algorithm to generate the necessary local rules for a desired global behaviour. Previous approaches are based on manual trial and error, for example, and expect the engineer to have a basic understanding of the system. Another alternative is often to copy existing systems in nature, which, however, requires the existence of a suitable example.

Current research aims to apply evolutionary algorithms to the design of a self-organising system. Another connection point is the link to cybernetics. The roots of cybernetics emerged in the 1940s, when commonalities between the brain and computers were investigated and intersections of various individual disciplines were identified, looking at human behaviour, message transmission, control engineering, decision and game theory and statistical mechanics. Towards the end of the winter of 1943/44, Norbert Wiener and John von Neumann organised a joint meeting in Princeton with engineers, neuroscientists and mathematicians on this topic. A further catalyst for this development were the Macy conferences from 1946 to 1948 with the theme "Circular causal, and feedback mechanisms in biological and social systems" and from 1949 to 1953 with the programmatic title "Cyber- netics".

The term was first used in print by Norbert Wiener in 1948 in Cybernetics or Control and Communication in the Animal and the Machine. In the same year, he published a fundamental review article on cybernetics in the journal Scientific American.

From 1948 onwards, John von Neumann introduced further additions to cybernetics in his lectures. The result of these thought experiments was the theory of self-reproducing automata or self-replication in 1953. These concepts transferred properties of genetic reproduction to social memes and living cells and, since the 1970s, to computer viruses. In 1961, Norbert Wiener supplemented his cybernetics basics book with two further chapters: On Learning and Self-Reproducing Machines and Brainwaves and Self-Organising Systems.

The philosopher and logician Georg Klaus established the subject of cybernetics at the Chair of Logic and Epistemology at the Humboldt University in Berlin in 1953. Later, he was involved in founding his own cybernetics commission at the Academy of Sciences of the GDR. Georg Klaus (1912 - 1974) was a German Marxist philosopher as well as a chess player and chess official. One of Georg Klaus' philosopher concerns was the connection of his philosophy with the modern sciences. He had recognised that there were considerable gaps in philosophical reception in this field. Marxist philosophy in the middle of the 20th century had great difficulties with a materialistic understanding of mathematics and logic, with newer results in physics (for example, on space and time) as well as with disciplines such as semiotics and cybernetics. This explains his intensive engagement with modern logic, cybernetics, semiotics and a general methodology of the sciences. His work in this area always included rejecting unscientific and dogmatic philosophical interpretations of scientific results.

8.2 The creation of chaos

The illustration below was created by not allowing the colours to be ordered. The requirement was to select two rows and two columns in this 18 x 18 grid and thereby determine the corner points of rectangles. The colouring of the elements with four colours should be done in such a way that in no rectangle the corner points have the same colour. This task has an electronic background, but it does not play a role here.

Very typical structures appear for the colours of the nodes; we consider the colour blue as an example.

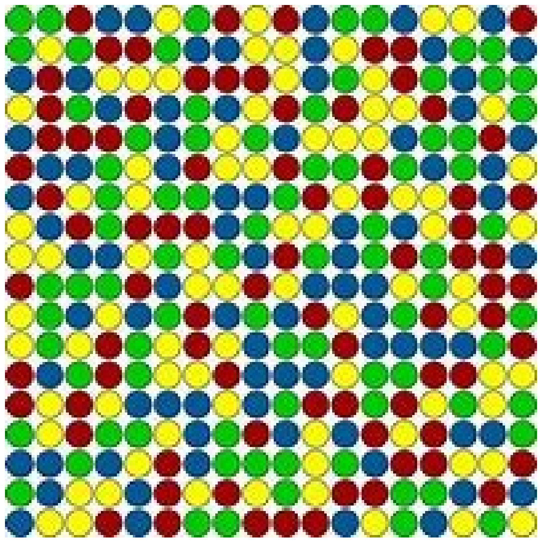


Figure 8.1: A four-colour grid

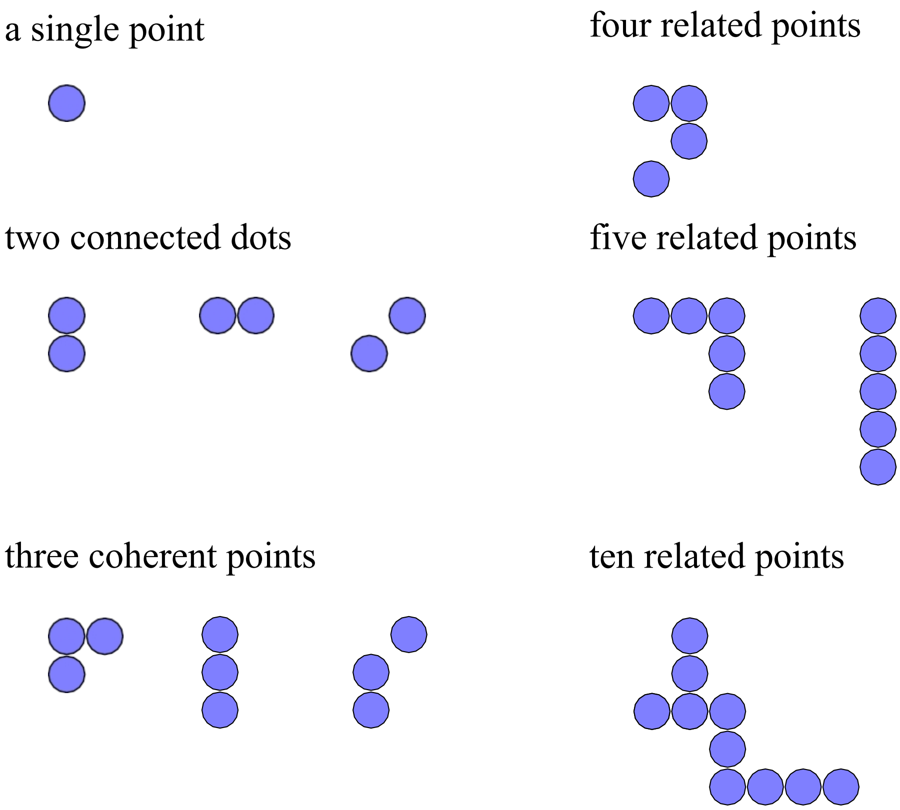


Figure 8.2: Combinations of blue dots

You can see that for some obscure reason the blue dots are oriented relatively straight, as are the red and green dots. The yellow dots, however, appear more as oblique shapes.

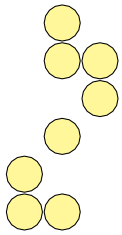


Figure 8.3: Combination of yellow dots

The fractal structures can possibly be used to simplify the calculations. The initial problem was solved with a logical equation whose solution space contained ^{4324} elements.

It is a current research problem whether one cannot proceed in the following way:

1. One solves the problem for a 6 x 6 grid and uses this solution along the main diagonals, three times in total. The complexity

2. One solves the problem for the next field 6 x 6 and uses the solution twice above the main diagonal. This does not increase the number of variables, only the number of constraints. This is rather convenient because it limits the number of solutions more quickly.

3. Do the same below the main diagonal.

4. For the corner squares, use the same procedure for the 6 x 6 square, but you can use the solution on the bottom left as well as on the top right.

If you go back, you can see that this representation can be understood, for example, as percolation structures or as stamens on a surface. If you were to sprinkle the four different colours in powder form over a square surface, you would certainly get a similar picture.

8.3 The butterfly effect

Another type of opaque relationship can be described under the heading of "The butterfly effect is a phenomenon of non-linear dynamics. The butterfly effect is a phenomenon of nonlinear dynamics. It occurs in non-linear dynamic, deterministic systems and manifests itself in the fact that it is not possible to predict how arbitrarily small changes in the initial conditions of the system will affect the development of the system in the long term.

The eponymous illustration of this effect using the example of weather comes from Edward N. Lorenz "Can the flap of a butterfly's wings in Brazil trigger a tornado in Texas?" The analogy is indeed reminiscent of the snowball effect, in which small effects amplify themselves via a chain reaction until a catastrophe occurs. The butterfly effect, however, is about the unpredictability of long-term effects.

Lorenz did preliminary work on the theory with a paper from 1963, in which he undertook a calculation for weather forecasting with the computer. In connection with long-term weather forecasts, he used a simplified convection model to investigate the behaviour of liquids or gases when they are heated: here, rolls initially form (hot gas rises on one side, loses heat and sinks again on the other side), which become unstable when further heat is added.

He characterised this behaviour using the three connected differential equations. He projected the numerical result into phase space and obtained the strange attractor that later became known as the Lorenz attractor: an infinitely long trajectory in three-dimensional space that does not intersect itself and, from a suitable angle of view, has the shape of two butterfly wings.

Lorenz came across the chaotic behaviour of his model rather by chance. In order to save computing time, he had used intermediate results from previous calculations for the numerical solution of the equations, but only took three decimal places into account, although the computer calculated with an accuracy of six decimal places. The result was increasing deviations in time between the old and the new calculations, which led Lorenz to his statements about the sensitivity to the initial conditions. From almost the same starting point, the weather curves diverged until they finally showed no commonality.

In his first calculation, he gave a starting value for an iteration on six de-

(0.506127), and in the second calculation to three (0.506), and although these values differed by only about 1 10,000, as time went on this calculation diverged greatly from the first.

The butterfly effect occurs in systems that show deterministic chaotic behaviour. These systems have the property that arbitrarily small differences in the initial conditions (clinamen) lead to strong differences in the system over time; they are therefore sensitively dependent on the initial values. This phenomenon can be quantified by means of the so-called Lyapunov exponents.

Some examples:

- Meteorology: Since the initial conditions can only ever be determined experimentally with finite accuracy, one consequence of this effect for such systems is that it is impossible to predict their behaviour for a longer period of time. For example, the weather can be predicted relatively accurately for one day, whereas a prediction for a month is hardly possible. Even if the entire surface of the earth were covered with sensors, if they were only slightly spaced apart, if they reached into the highest layers of the earth's atmosphere and if they provided exact data, even a computer with unlimited performance would not be able to make exact long-term forecasts of weather development. Since the computer model does not cover the spaces between the sensors, there are minor divergences between the model and reality, which are then positively amplified and lead to large differences.

  For example, the data from 1000 weather stations can be used to make reasonably reliable forecasts over a period of four days. For corresponding forecasts over eleven days, one would need 100 million measuring stations evenly distributed over the earth. The project becomes absurd if the forecast is to extend over a month; then one weather station would be needed for every five square millimetres of the earth's surface.

  However, the Lorenz model is actually much more chaotic than the actual weather pattern. The equations are much more unstable than the basic physical equations. The mathematician Vladimir Igorevich Arnold gives two weeks as a basic upper bound for weather forecasting.

- Planetary orbits: If more than two celestial bodies are gravitationally bound to each other, minimal changes in the initial situation over time can lead to large unpredictable changes in the orbits and positions of the planets.

lead. This behaviour is the subject of the three-body problem.

The Lorenz attractor has already been shown in Fig. 6.2, the corresponding slides are given in 6.3.

Figure 8.4: The butterfly effect (Valeria Sabater)

The three-body problem shows the same sensitivity to the initial conditions. It consists of finding a solution (prediction) for the course of the orbit of three bodies under the influence of their mutual attraction (Newton's law of gravitation). In order to obtain quantitative results, it must be solved numerically in the general case so far.

The three-body problem has been considered one of the most difficult mathematical problems since the discoveries of Johannes Kepler and Nicolaus Copernicus, and many famous mathematicians such as Alexis-Claude Clairaut, Leonhard Euler, Joseph-Louis Lagrange, Thorvald Nicolai Thiele, George William Hill and Henri Poincaré have dealt with it over the centuries. In the general case, the movement is chaotic and can only be calculated numerically.

The special case in which one of the three bodies has a vanishingly small mass and its effect on the other two can be neglected is called the restricted three-body problem. It plays an important role in astronomy (e.g. in research satellites such as the Planetary Grand Tour), which leads to the problem of Lagrange points.

The two-body problem can be solved analytically by Kepler's laws. In contrast, the integrals in the case of more than two celestial bodies are no longer algebraic integrals and can no longer be solved with elementary functions. At the beginning of the 20th century, Karl Frithiof Sundman was the first to give an analytical solution of the three-celestial body problem in the form of a convergent power series, under the assumption, that the total angular momentum of the system does not disappear and therefore a triple impact does not occur in which the distance of all three bodies is zero. For practical calculations, however, Sundman's solution is not useful, since at least 10 to the power of 8,000,000 terms would have to be taken into account in the sum to achieve sufficient accuracy.

The stability of a three-body system is described by the Kolmogorov-Arnold-Moser theorem: if an unperturbed system is not degenerate, then for sufficiently small autonomous Hamiltonian perturbations most of the non-resonant tori are only slightly deformed, so that invariant tori also exist in the phase space of the perturbed system, which are densely and quasi-periodically spun around by the phase orbits, the frequencies being rationally independent. These invariant tori form the majority in the sense that the measure of the complement of their union is small when the perturbation is weak.

Approximate or exact solutions are possible in some cases: If the mass of one of the celestial bodies is small, then the three-body problem is solved iteratively, nowadays with computers, or orbital perturbations are calculated that the smallest (lightest) body suffers from the larger (heavier) ones. The special case of the equilibrium of the attractive force of two large (heavy) bodies on a vanishingly small (light) body (taking into account the apparent forces occurring in the rotating reference frame) can be solved exactly.

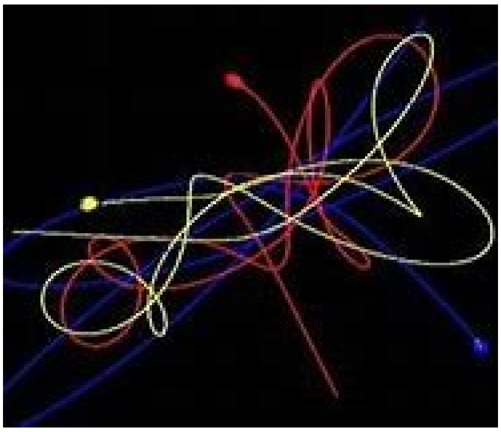


Figure 8.5: The three-body problem

Another very chaotic behaviour can be described with the keyword "herd instinct". It was already mentioned in the chapter on "Fractals in human society". However, it will now be presented in a much broader way.

In zoology, a herd is predominantly a collection of large, usually similar, often exclusively herbivorous amniotes, especially large mammals and large ratites. The term is independent of whether there are

The term "herd animal" is used to refer to both wild and domesticated ungulates. In particular, ungulates living together in herds, both wild and domesticated, are referred to as herd animals.

Figure 8.6: A wildebeest herd in Tanzania

A herd is a more or less uniformly coordinated social association of less than ten to several thousand individuals. Depending on its size, a herd can be an anonymous social group in which most individuals do not know each other, or an individualised social group in which the animals are familiar with each other. Under certain circumstances, especially in ruminants, smaller groups in which the group members have closer ties to each other unite to form large anonymous herds. Such large herds can then also be composed of animals of different species, for example wildebeest, zebra and ostrich.

Smaller herds can either be loosely organised without a leader, as in male deer outside the mating season, or hierarchical with a leader or alpha animal, as in horses. Herd behaviour depends on many factors, be it food availability or species-specific reproductive behaviour. A large herd with many alert animals reduces the likelihood of an individual animal being preyed upon by a predator. Penguins hibernate in large numbers close together, which reduces the loss of body heat. Herd behaviour is considered an evolutionary adaptation.

We return once again to the topic of chapter 4 and look at the behaviour of crowds from the point of view of chaos theory. Mass psychology is a branch of social psychology and deals with the behaviour of people in crowds. The starting point for the development of the theory of crowd psychology is the fact, which is part of general experience, that large crowds of people often show behaviour that appears surprising and irrational, for example the triggering of panic due to a rather insignificant occasion.

Important decisions in a group are not made by single individuals.

The masses do not take decisions, but bring them about through a vote in order to achieve a goal by working together. Throughout history, large masses of people have been able to initiate dramatic and sudden social change outside of established legal processes. Collective cooperation is condemned by some, supported by others. Social scientists have put forward several different theories to explain mass psychological phenomena and how group behaviour differs significantly from the behaviour of individuals within the group.

- Contagion theory: an early theory of collective behaviour was formulated by the French sociologist Gustave Le Bon in his major work Psychology of the Masses (1895). According to Le Bon's contagion theory, social groups exert a hypnotic effect on their members. Protected in the anonymity of the crowd, people give up their personal responsibility and surrender to the contagious feelings of the masses. The crowd thus develops a life of its own, stirs up emotions and tends to lead people to act irrationally. However, as Clark McPhail points out, systematic research reveals that "the mad crowd" does not have a life of its own separate from the thoughts and intentions of its members. Norris Johnson, who researched a panic during a Who concert in 1979, came to the conclusion that the crowd consisted of many small groups whose members predominantly tried to help each other.

  Le Bon's work is the starting point of Sigmund Freud's study "Mass Psychology and Ego Analysis", but Le Bon was working on "unstable, gathering masses, while Freud was analysing more highly organised, stable masses such as the church and the army".

  Wilhelm Reich formulated his work "The Mass Psychology of Fascism" from his own further development of psycho-analysis in 1933; Elias Ca- netti substantiates this thesis in his more literary work "Mass and Power".

- Convergence theory: Convergence theory postulates that mass behaviour does not originate from the masses themselves, but is brought into the group by single individuals. Group formation itself amounts to the convergence of individuals with similar dispositions. In other words, the contagion theory says that groups cause people to act in a certain way; the convergence theory, on the other hand, says the opposite: people who want to act in a certain way join together.

An example of the convergence theory is a phenomenon that can sometimes be observed when more immigrants appear in a previously homogeneous area and members of the existing community (apparently spontaneously) band together to threaten the newcomers. Supporters of convergence theory believe that in such cases it is not the mass that generates racial hatred or violence, but that hostility has been simmering in many residents for a long time. The mass arises from the convergence of those people who are against the new neighbours. The theory of convergence states that the behaviour of the masses is not itself irrational, rather the people express views and values that exist in the group, so that the reaction of the mob is only the rational product of widely dispersed popular sentiments.

- Leadership theory: not necessarily, but often in Le Bon's (and Gabriel Tarde's) case, masses are led by self-elected - or at least collectively recognised - leaders and are sometimes seduced into acts that they would probably not commit outside the mass, as individuals. A special case of such mass leadership exists when the leader knows how to draw to himself the masses' concentrated community solidarity and thus also their powerful sense of self-worth. The leader then "embodies" the masses, their goals and values, their thinking and their emotions; he presents himself as their "supreme die- ner" and only becomes their master through this apparent subjugation. This can lead to the point where the extraordinarily pronounced mutual sympathy that the members of the masses had for each other is now concentrated more and more on t h e leader: the masses begin to love their leader.

  This is where the term charismatic mass leadership is used in social psychology. The glorification of leaders leads to the attribution of special "ingenious", "miraculous", almost "divine qualities", which make the leader particularly capable of leading the masses and which make it possible for the masses to be led by him.
  The "blind trust" that is placed in him is justified. Imaginative legends, anecdotes and rumours of the masses, but also targeted mass propaganda of the leader and his followers confirm and consolidate such valuable attributions. The masses begin to believe in their leader as in a figure of salvation. Blind mass obedience, even "to the death", is sometimes voluntarily given to the leader and, once established, demanded of him as a matter of course. Charismatic mass leadership of this ex- treme form, which occurs most frequently in the case of religious, military and political leadership, is favoured by the "messianism of the masses" (Michael Günther), a form of the particularly pronounced religiosity of the masses, which is already mentioned by Le Bon.

This messianism creates a power vacuum, a "charismatic gap": the masses develop a desire for clarity and leadership, especially in the presence of strong collective emotions (such as fear of death or supreme confusion) and a low level of horizontal organisation. Their will to survive is concentrated on the hope of gifted leadership, all the stronger the more desperate their situation appears. If their self-esteem is so damaged, if the masses are convinced that they can no longer extricate themselves from a hopeless situation by their own efforts, they are also ready for subordination, for the recognition of a higher value than their own. The bearer of hope who skilfully exploits the charismatic gap is welcomed like a messiah sent by the Schick- sal. Rescue from adversity seems within reach, joy and relief spread, gratitude is expressed to the one who embodies the "last hope". The leader skilfully exploits the suggestibility of the highly emotionalised masses and reinforces his otherworldly nimbus by seizing on the collective hopes of the masses and presenting himself as a messiah sent by higher powers to fulfil a specific mission.

However, the complex interplay of mass hopes and the charismatic leader's chances of leadership only works as long as the leadership figure proves himself: if everything fails, if the charismatic leader all too obviously disappoints the exuberant hopes of the masses, the legitimacy he had gained as a mass leader is quickly withdrawn, the masses no longer follow and obey him and withdraw their affection. The instability of charismatic mass power lies in the possibility of the charisma's rapid disenchantment if it fails to prove itself. This is another reason why charismatic mass leaders often try to combine their power with rational and traditional ruling psychology, which guarantees more stability and can also help them overcome defeats and blows of fate - which charismatic mass power could hardly survive on its own. Paradigms of such entanglements of charismatic mass leadership with rational and traditional psychology of rule can be found throughout history: Alexander the Great provides an example, as do Gaius Julius Caesar and Napoleon Bonaparte. The 20th century brought a particularly large number of charismatic mass leaders to power, such as Benito Mussolini, Vladimir Il- jitsch Lenin, Josef Stalin, Adolf Hitler, Mao Tse-Tung and numerous lesser-known leaders.

- Areas of application: In addition to politics, the financial market is an important area of application in which mass psychology research can be established. The combination of knowledge about investor behaviour with the findings of mass psychology reveals new models and approaches.

This is because the cyclical nature of booms and depressions is a recurring element in financial market history. The boom and bust cycle is a recurring element in financial market history and traditional economic theories and financial market models (e.g. market efficiency hypothesis) fail to explain and predict such trends and the underlying behaviour of market participants. This is because they do not take into account the whole human being, but only an academic abstraction of those aspects of human behaviour that they consider to be economically relevant. And they also forget society, with which markets are inextricably linked.

This is where mass psychology, which is based, among other things, on the concepts of mutual social and psychological contagion as well as the human tendency to orient and imitate others in the social environment, comes in. The study of collective dynamics provides yet another contribution to a better understanding of the processes in the financial markets by pointing to the connection between short-term developments and long-term processes of change. Based on the principle of long and short cycles, collective psychology distinguishes between consciously recognised, short-lived effects and their underlying slow, sub- tle and often unrecognised developments. Through this central insight, which essentially goes back to Gustave LeBon, mass psychology research contributes to the description of the interaction of a mountain of debt accumulated since the 1960s and the periodic emergence of boom-crisis cycles during the past decades.

- The Wisdom of Crowds: the Wisdom of the Many - Why Groups are Smarter than the Few and How Collective Wisdom Shapes Business, Economies, Societies and Nations. Why the Many Are Smarter than the Few and How Collective Wisdom Shapes Business, Economies, Societies and Nations") is the title of a book by James Surowiecki published in 2004. He argues that the accumulation of information in groups leads to joint group decisions that are often better than the solutions of individual participants (so-called collective intelligence).

  The book presents numerous case studies and anecdotes to illustrate its argument. Many disciplines are touched upon, but mainly economics and psychology.

  The introductory story tells of Francis Galton's surprise that visitors to the West of England livestock fair in 1906 estimated the slaughter weight of a cow extremely accurately as part of a competition, if the median of all 787 estimates was taken as the group's estimate. The mean value of the

The individual estimates were even accurate and better than those of any of the participants, including some experts such as Metzger.

The book refers to different groups of independently deciding people, not to phenomena of mass psychology. He draws parallels to statistical selection procedures, according to which a different group of individually deciding people can rather represent the totality of all possible outcomes of an event and is thus able to make better predictions for the future. Surowiecki divides decisions into three main groups, which he classifies as problem areas:

– Cognition: this problem area includes decisions where there is a concrete solution that can be identified through the use of cognitive skills. Surowiecki argues that a group can do this much more accurately, quickly and independently of political forces than experts or groups of experts.

– Coordination: Coordination of behaviour includes optimising the use of a restaurant or driving accident-free. The book contains many examples from experimental economics, but this section is based more on naturally occurring phenomena, such as pedestrians optimising pavement use or the occupancy of popular restaurants. It explores how shared beliefs/norms within a culture allow for surprisingly accurate predictions about the reactions of other members of that culture.

– Cooperation: how groups of people can build a network of trust without needing central control over their behaviour or direct enforcement of the rules. This section makes a particular case for a free market.

Not all groups are wise. Examples of such reasoning include angry crowds or investors in the stock market after a boom or crash. Research is needed to uncover and avoid more examples of faulty group intelligence. Nevertheless, it is possible to define key criteria that distinguish a wise group from an irrational group.

- Diversity of opinion: Every person has different information about an issue, so that there can always be individual interpretations of an issue.

- Independence: The opinion of the individual is not determined by the view of the group.

- Decentralisation: Here the focus is on specialisation in order to apply the knowledge of the individual.

- Aggregation: Mechanisms are in place to form a group opinion from individual opinions.

  Surowiecki looked at situations where the group built up a very bad reputation and argued that in these situations the knowledge or cooperation was faulty. In his view, this happened because the group members listened too much to other people's views and emulated them instead of making up their own minds about the situation and differentiating. He gives various details of experiments according to which group habits become known through a selected speaker. On top of that, he claims that the main reason for the intellectual conformity of a group is mainly to make systematic wrong decisions.

  According to Surowiecki, if the deciding authority is not able to accept the group, this leads to the loss of the personal right and the right to self-information. In this way, the cooperation in the group can only be as good, or rather worse than better, than the smartest member (the possibility exists on the face of it). Detailed case studies include the following errors:

- Centralism: The accident of the space shuttle Columbia, whose culpability shifted to the bureaucratic hierarchy of NASA management, claiming not to have known about the engineers' warnings.

- Differences of opinion: The US community could not prevent the at- tent of 11 September 2001 because information from one sub-agency was probably not passed on to another. According to Surowiecki, groups work best when they self-select their work and obtain information they need themselves (in this case IQ researchers). The isolation of the SARS virus serves as an example of the impossibility of coordinating research. He interprets the isolation of the virus as an example of the free flow of data to coordinate research, through laboratories around the world without a central control point.

- Ambivalence: Where transitions become visible and are portrayed in a slowed down way.

> If the group is not independent, there can be a flood of information that the decisive individuals do not notice, taking into account the choice made: provided this happens, it is easier for the individual to adapt his behaviour to the group, as he can easily copy the behaviour of the group. Loss of independence in the group

Herd behaviour is also found in human society, for example when visiting restaurants or theatres, in fashion or when buying bestsellers. In financial markets, investors sometimes tend to behave like a herd in their buying and selling decisions and to invest or disinvest in a trading object. Herd behaviour is a manifestation of mass psychological contagion effects and can thus be a cause of financial market crises or economic crises. Hoarding purchases also show herd behaviour, such as before natural disasters or during the Covid 19 pandemic from March 2020, when there were shelf gaps for certain goods (e.g. flour, pasta, toilet paper) in German shops.

One of the most important researchers on herd behaviour is Abhijit Banerjee, who defined herd behaviour in 1992 as decisions made in the same direction based on different private information. He implied an asymmetric information, which consists in the fact that poorly informed market participants see a profit opportunity or risk of loss in the market behaviour of the better informed market participants and imitate their behaviour.

Herd behaviour is therefore due to the social effect of imitation. It exists when an individual decision-maker, taking into account the behaviour of other actors, decides to behave in the same way as them, even if his own independent decision would turn out differently.

In addition to these information cascades, there is also the reputation model in the context of the beauty contest and network effects. As early as 1936, John Maynard Keynes examined herd behaviour in the context of the beauty contest in investment decisions. This is based on the assumption that managers follow the behaviour of other managers for reasons of reputation in markets with imperfect information. This beauty contest leads to financial analysts forecasting not the expected earnings power of companies, but what the majority of all analysts will forecast. Any forecast errors are amplified as a result. A network effect exists if there is a certain cause-and-effect relationship between the action of one market participant and the actions of the others. The benefit for the individual market participant increases if other market participants decide in favour of the same product.

differentiate. For example, in social networks such as Facebook, their usefulness increases the more users this online community has.

Herd behaviour can be based on various mass-psychological or market-psychological causes. The consumer may be driven by the fear of not being able to meet his demand in the face of shelf gaps if he does not buy immediately. A consumer's expectation that other consumers will also hoard after him also pushes him to hoard purchases. Likewise, his fear that there might be supply bottlenecks in the future forces him to make purchasing decisions that are not in line with his needs. Sometimes consumers' feelings of powerlessness are also seen as a cause.

The behaviour is irrational, especially since food and luxury goods or toilet paper, as mass products, can be reproduced at any time. In France and Italy, one of the products affected by hoarding (achat de ravitaillement in French, acaparmiento in Italian) is red wine, a product that cannot be reproduced at any time. In Turkey, "Kolonya", which is similar to cologne, has become scarce due to hoarding.

However, it is difficult to prove herd behaviour explicitly; a joint purchase/sale of a certain security by many economic agents need not necessarily be due to herd behaviour (and thus to information asymmetries); it may also be a matter of chance. If new information makes the current price of the security appear wrong and if this information becomes known to many economic agents at the same time (perfect information), this can result in many simultaneous sell decisions made uninfluenced by the behaviour of other investors. A specific type of herd behaviour is noise trading, when "noise traders" are motivated by the existing noise to buy into rising prices (bull market) or sell into falling prices (bear market).

The consequence of herd behaviour is strong price fluctuations of the trade object concerned. In addition, hoarding accelerates the rotation of goods and reduces the logistical reach.

Herd behaviour is known as market behaviour, especially among noise traders, who are often guided by herd behaviour and motivated by sentiment or groups to buy or sell into falling ones. This is the so-called "sentiment noise". Rising or falling prices are an indication that other market participants have already made the same decision before. This noise can underlie both buy and sell decisions and also hold decisions. Herd behaviour is therefore a sign of a lack of

Efficiency of markets.

Speculation only becomes problematic for a market when it is no longer speculated with the help of fundamental data, but herd behaviour sets in. Then speculative bubbles can arise, which are usually due to herd behaviour. Speculative bubbles can be justified by the expectation of the majority of market participants of future profit opportunities.

Profit-taking can also be based on herd behaviour, when a large number of investors take advantage of a high price level to sell and are joined by other investors. The bank run is also a typical herd behaviour, as investors observe a perhaps random mass withdrawal of cash and blindly join it, trusting that it must have a specific reason; the mass withdrawals eventually culminate in the domino effect. Investors withdraw their deposit because they fear that they will not be able to withdraw it otherwise due to the sequential payout principle ("first come, first served") as the cash reserves are depleted. Consequently, it is rational for every depositor to follow the herd. A bank run is more likely the less informed bank customers are and the more they "overreact". Hoarding is likely to contribute to the scarcity of certain goods or services and thus to market tightness by strongly increasing demand.

Herd behaviour can lead to self-fulfilling prophecies: If market participants behave in a certain way, this can lead to the underlying fundamentals of an investment changing as a result of the herd behaviour itself: they develop in the direction the herd is taking - consequently, it is rational not to break away from the herd, which ultimately leads to the expected result.

Mass hysteria refers to a strong emotional excitement in large crowds, for example (euphoric) on the occasion of rock and pop concerts, major sporting events or (mourning) after the death of famous people. The term is to be distinguished from moral panic, which is specifically used for social control.

This usage dates back to The Quarterly Christian Spectator in 1830 and was used, among other things, during an outbreak of cholera. Marshall McLuhan began to describe the phenomenon scientifically in Understanding Media in 1964.

In this sense, the excessive enthusiasm for the Beatles, for example, was and is assigned to the realm of mass hysteria, as is the mourning for Rudolph Valentino, Josef Stalin or Eva Perón. The medieval dance mania, the witch craze of the early modern period and other mass fears (for example communist fear in McCarthyism) are often described as mass hysteria. The term is also occasionally used as synonymous with mass panic. Social psychology deals scientifically with the behaviour of people in crowds under the topic of mass psychology.

The term mass panic refers to an accident involving a large number of people in a confined space, in which the spatial confinement is one of the reasons for the course of the accident. It suggests the idea that a mass of people panics at large events or incidents and that uncontrolled escape movements occur. The cause of a mass panic can be dangerous external circumstances (such as a fire or the collapse of a building) or the behaviour of individuals within a crowd. The terms mass accident, mass casualty and mass panic are often used synonymously in the media. A mass panic only occurs in the course of a very small proportion of mass accidents.

Heavy crowding or disasters involving many people can trigger mass panic, accompanied by uncontrolled fear and massive flight movements. In such a situation, there are only a few possibilities for intervention. The g r e a t e s t possibilities of influence exist in the development phase or before. Targeted, clear, frequent, regular and structured calls and information are important. This can be done, for example, through loudspeaker announcements or through procedures that demonstrate composure (e.g. continuation of the event like a football match). Attention-getting interventions (e.g. a shrill whistle) or setting simple tasks can also reach a panicked crowd (e.g.: Watch out for children!).

It has certainly become clear that all these described structures have a fractal character. It is always a matter of a large quantity of individuals crowded into a small space. These quantities can behave both positively and negatively. As an individual, one is always well advised (as in avoiding Corvid - diseases) to keep a reasonable distance from other individuals. The distance must be large enough to prevent the spread of negative factors, but it must also be small enough to allow the spread of positive phenomena.

Let us return to the animal world. Fish, birds, grasshoppers, mosquitoes, ants, bees - here you will find any number of other fractal, chaotic, self-organising structures.

Figure 8.7: Ants rarely appear alone

Figure 8.8: A swarm of locusts Finally, a chaos

that is surely familiar to many.

8.4 Chaos in the Nature

Earthquakes are associated with fractals and chaos in two ways. First of all, the recordings of vibrations caused by an earthquake have a fractal character. And the fact that chaotic conditions can prevail after an earthquake is sufficiently well known from literature, television and films.

Storms occur in several gradations:

- A wind is considered a storm from a wind speed of around 75 kilometres per hour.

Figure 8.9: A car traffic jam

- In a broader sense, hurricanes are winds with a force of 12 on the Beaufort scale; in a narrower sense, they are North Atlantic lows in which such winds with a force of 12 occur. In the past, all winds with hurricane force were referred to as hurricanes.

- Tornadoes are the most devastating form of storms. A tornado is a small-scale air vortex in the earth's atmosphere with an almost vertical axis of rotation. It is associated with convective cloud cover (cumulus and cumulonimbus) and thus differs from small tornadoes (dust devils). The vortex extends continuously from the ground to the cloud base, but does not have to be condensed throughout. This definition goes back to Alfred Wegener (1917) and is still generally accepted today.

  For a tornado to develop, the conditions for high-level moisture convection must first be met. These are conditional lability, i.e. a sufficiently strong vertical temperature decrease, sufficient moisture supply (latent heat) in the lower 1-2 km of the atmosphere as well as uplift of the air mass to trigger the moisture convection. Lifting mechanisms can be thermal (solar radiation) or dynamic (fronts). The main energy supplier of such storms and thunderstorms in general is the latent heat stored in the water vapour of the moist air mass, which is released during condensation. It is this additional heat that enables the air to rise freely to a high degree (moisture convection), since the atmosphere is not susceptible to dry convection, apart from ground-level convection.

Figure 8.10: The consequences of a major earthquake in Haiti

overheating, is stable. In the latter case, only the formation of small currents can occur. Small tornadoes, the so-called gust front vortices or gustnadoes, can form on the gust front of a shower or thunderstorm. These can develop into tornadoes if they come into contact with the moist convective updraft and are thus intensified.

In the initial stage, a tornado is almost invisible. Only when water vapour condenses or dust, debris, water and the like are whirled up inside the vortex due to the pressure drop and the resulting adiabatic cooling, does the tornado also appear visually. Continuous condensation from the cloud to the ground is not always observed. Such condensation originating from the parent cloud is called a funnel cloud. If the air vortex does not reach the ground, it is called a blind tornado. For a tornado, the contact of the vortex with the ground is decisive, not its continuous visibility. If, for example, wind effects are detectable under a funnel cloud, i.e. damage to the ground as a rule, it is a tornado. The shape of the vortex is very diverse and ranges from thin tube-like forms to a more or less wide funnel widening upwards (see adjacent illustrations and web links). The diameter can range from a few metres to 500 m and even more than 1 km. It is not uncommon - especially with large diameters - for several vortices to occur that circle around a common centre, which is called a multivortex tornado. Dust, debris and condensed water can sometimes prevent a multivortex tornado from being recognised as such because the individual vortices are not visible.

The force of a tornado can cause a wide range of damage. It can damage houses

Figure 8.11: The eye of a hurricane

and cars and poses a danger to animals and people. Even stone houses are not safe. Indirect damage is caused by flying debris. The main cause of damage is the dynamic pressure of the wind and, above about 300 km/h, also increasingly indirect damage caused by flying debris. The earlier assumption that the strong negative pressure within a tornado, which can be up to 100 hPa, causes buildings to explode, as it were, is no longer tenable. Due to their high wind speeds, which change in a small area, tornadoes are in principle a danger to air traffic; however, accidents are rare due to the small-scale nature of this weather phenomenon. A spectacular case occurred on 6 October 1981, when NLM Cityhopper Flight 431 was caught in a tornado and crashed after the right wing tore off. All 17 people on board died.

The duration of a tornado ranges from a few seconds to more than an hour, on average less than ten minutes. The forward motion of a tornado follows the associated parent cloud and is on average around 50 km/h, but can also be significantly lower (practically stationary, not uncommon with waterspouts) or higher (up to over 100 km/h with strong upper-level flow). The tornado track is essentially linear with minor deviations caused by the orography and the local wind field in the vicinity of the thunderstorm cell.

However, the internal rotational speed of the wind is usually much higher than that of the linear motion and is responsible for the severe devasta-

The highest wind speed ever recorded within a tornado was during the Oklahoma tornado outbreak. The highest wind speed ever recorded within a tornado was determined with a Doppler radar during the Oklahoma tornado outbreak on 3 May 1999 near Bridge Creek, Oklahoma (USA). At 496 ± 33 km/h, it was in the upper range of the F5 class of the Fujita scale; the upper error limit even extends into the F6 range. This is the highest wind speed ever measured on the Earth's surface. Above the earth's surface, only jet streams reached higher wind speeds. In the official statistics, however, this tornado falls below F5 with regard to the most probable value and the uncertainties.

In the US, about 88 per cent of observed tornadoes are weak (F0, F1), 11 per cent are strong (F2, F3) and less than 1 per cent are devastating (F4, F5). This distribution function is very similar worldwide and in this form dominated by mesocyclonic tornadoes, which fill the full intensity spectrum. In contrast, the intensity of non-mesocyclonic tornadoes hardly exceeds F2.

Tornadoes form most frequently over land in early summer, with the maximum occurring later with increasing latitudes. Over water, the maximum is reached in late summer, because that is when the water temperature and consequently the lability is highest. The same applies to the diurnal cycle. Tornadoes over land are most likely to occur in the early evening hours, while the maximum for waterspouts is in the morning hours. Furthermore, waterspouts show a climatological difference in the annual cycle, depending on whether they move ashore or remain over the water. The seasonal distribution for the first case is similar to that for tornadoes over land, while pure waterspouts show the aforementioned late summer maximum.

Tornadoes are observed worldwide wherever there are thunderstorms. The focal points are regions with fertile plains in the subtropics up to moderate latitudes. The most frequent region is the Midwest of the USA, where the climatic conditions for the formation of severe thunderstorms and supercells are very favourable due to the wide plains (Great Plains) east of a high mountain range (Rocky Mountains) and north of a tropical sea (Gulf of Mexico). For weather situations with a high storm potential, the mountains cause relatively dry and cool air masses in the middle to upper area of the troposphere with southwesterly to westerly winds, while in the lower layers warm and humid air masses from the Gulf region can be transported northwards without hindrance. This results in an unstable stratification of the atmosphere with a large supply of latent heat and a directional wind shear.

Other important regions are Central, Southern and Eastern Europe, Argentina, South Africa, Bengal, Japan and Australia. Numerous, albeit on average weaker, mostly non-mesocyclonic tornadoes occur in the area of the Front Range (eastern edge of the Rocky Mountains), in Florida and over the British Isles.

About 1200 tornadoes are registered annually in the USA, most of them in Texas, Oklahoma, Kansas and Nebraska along Tornado Alley with about 500 to 600 cases per year. This is due to the above-mentioned special climatic conditions, which provide the prerequisites for the formation of mesocyclonic tornadoes in particular far more frequently than in other regions. In addition, there are several regional clusters in the USA, e.g. in New England and Central Florida.

In Europe, the annual number of tornado observations is 330, of which 160 are over water, with an estimated 590 tor- nados, including an estimated 290 waterspouts, when the number of unreported events is included (2020: 800 reported events). As in the US, most European tornadoes are weak. Devastating tornadoes are rare, but eight F4 and two F5 events have been documented from Germany so far. The latter were already described by Alfred Wegener in 1917 in a paper on the tornado climatology of Europe. Other devastating cases are known from northern France, the Benelux countries, Austria, northern Italy and Switzerland.

In Germany, the number of tornadoes observed each year averages between 30 and 60, with a high number of unreported events, especially weaker ones. Exact figures are only available on the Tornadoliste.de. According to the figures currently available, about five or more F2s can be expected every year, one F3 every two to three years and one F4 every 20 to 30 years. According to current knowledge, an F5 is a once-in-a-century event or even rarer. In terms of area, Germany has as many and as strong tornadoes as Texas in the USA.

An overview of the spatial and temporal distribution of tornadoes within Germany and their intensity can be found in the web links. In general, the risk of tornadoes is highest in the west of the North German Plain.

On average, about three tornadoes per year have been observed in Austria over the past 30 years. However, since 2002, an average number of about five tornadoes per year has been observed due to the increased spotter and statistical activities, especially of volunteers. Taking into account a possibly high number of unreported cases as well as the still very underrepresented F0 cases, it is possible that the number of tornadoes per year has risen.

the actual, averaged, annual number is up to ten tornadoes.

Several F0 and F1 cases occur every year. On average, one F2 can be expected annually, or once every two years, and one F3 every five to ten years. So far, one F4 tornado has also been documented in Austria.

The highest tornado density can be observed in south-eastern Styria (around three tornadoes/10,000 km^2 / year), followed by the area around the Hausruck in Upper Austria, the Vienna Basin, the region around Linz, the western Weinviertel, the Klagenfurt Basin, Lake Constance region and the Inn Valley in the area of Innsbruck.

Figure 8.12: A tornado in Oklahoma

In general, the occurrence of tornadoes is subject to strong fluctuations, which manifests itself in clusters within quite short periods of time, followed by quite long periods of relative calm. The outbreaks are due to the close connection with certain weather conditions, where several factors for tornado formation come together (see above under formation). Larger events of this kind with devastating tornadoes are known mainly from the USA. For Western and Central Europe, the years 1925, 1927 and 1967 are worth mentioning, with a focus on Northern France/Benelux/Northwest Germany. This region can also be considered a European tornado alley. The numerically most significant outbreak in Europe with a total of 105, but mostly weaker tornadoes (max. F2) hit the British Isles on 23 November 1981. At present, the data basis for Central Europe does not allow us to say whether tornadoes are occurring more frequently due to global warming, as the increase in the number of observed cases is mainly due to better recording in recent years.

- Floods, whirlpools and turbulence

A flood is a condition in which a normally dry ground surface is completely covered by water. Alongside particularly strong earthquakes, flood disasters have been the natural disasters with the greatest consequences for humans. Floods are mostly natural events.

Storm surges push seawater over river mouths, where there are no barriers, deep inland and over dikes on the coast and on coastal river sections, or cause dike breaches; tsunamis can also flood high coastal sections not protected by dikes.

Inland waters overflow their banks during floods if the water does not flow downhill quickly enough (as surface water downstream or seeping into the groundwater on unsealed soils). Floods may be caused by heavy rainfall, but also by the breach of dams or barrages due to excessive water pressure. Glaciers prevent water from flowing downstream and thus form an ice reservoir. Filling groundwater reservoirs lead to rising water levels if the groundwater lies above impermeable soil layers. It then finds its way into low-lying parts of structures from below or (below the earth's surface) from the sides. This mainly affects areas with a constantly high groundwater level. The inadequate or even non-existent drainage of large water masses in inland areas and on islands may be (at least partly) caused by humans:

The sealing of large areas makes it difficult for surface water to seep away and enter streams and rivers unhindered. However, heavy rainfall itself causes the sealing of low-lying areas, as it prevents the ability to absorb water even in soils that are not sealed per se from a certain point onwards. The straightening and narrowing of rivers leads to an increase in the flow velocity of the water, which quickly reaches lower-lying areas and runs into slower-flowing water at bottlenecks. Especially in urban areas, there are too few areas where the water can spread. This inevitably leads to flooding when there is a strong influx of water due to heavy precipitation and/or large amounts of surface water coming downstream.

Collapsed structures or flotsam can become jammed at bridges, barrages, rakes or spillways, or outlets, and thus cause waterlogging; even at great distances from larger rivers, (mostly localised) flooding can occur.

In most cases, heavy rainfall ensures that the sewage system and drainage ditches cannot cope with the water masses. Facilities created for the temporary storage of water masses, such as rainwater retention basins or planned floodplains, often prove to be undersized. In the cases mentioned, one of the main reasons for the occurrence of floods is that the probability of occurrence and the impact of heavy rain events are underestimated. Although water floods are a force of nature, a flood is not a natural event if it is caused by a burst water pipe or if areas are intentionally flooded (especially in connection with combat operations).

Floods can, under certain circumstances, cause considerable water damage to people's property and endanger the health and lives of people and livestock. If such a danger exists, emergency services speak of a water emergency. A distinction must be made between temporary floods, which end by draining off or pumping up the water that has penetrated, and permanent floods. The latter threaten low-lying coastal areas in particular as a result of climate-related sea-level rise.

In the case of flooding in mountainous terrain, damage can also be triggered below the actual water level by undermining properties on slopes. The material erosion can cause the slope above the water level to slide and buildings or their contents (e.g. a car in a garage) to fall down.

However, not every flood is a problem for humans. In desert areas, for example, there would be no fertile land for agriculture and horticulture without the regular overflowing of rivers such as the Nile away from oases, and from an ecological point of view, natural wetlands that are regularly flooded are considered extremely valuable.

In Germany, too, there are cultural landscapes whose inhabitants owe their prosperity to the fertility of their land, which in turn can be explained by sediments left on the ground after floods. This is the case, for example, in Artland, which is largely located in the Hase inland delta.

Whirlpools are mostly harmless.

Turbulent flow is the movement of fluids in which turbulences

Figure 8.13: A flood in Japan

Figure 8.14: Small whirlpools are pretty and harmless

occur over a wide range of size scales. This flow form is characterised by a three-dimensional flow field with an apparently randomly varying component in time and space. The spatial aspect is illustrated in the picture opposite, the temporal aspect, for example, by the noise of the wind.

Turbulence leads to increased mixing and, as a result, to effectively increased diffusion coefficients. In the case of large-scale turbulence, the contribution of molecular diffusion is negligible. The mixing also affects the internal energy (heat transport) and momentum.

The pressure loss of a fluid flowing through a pipe is due to the difference of momentum to the pipe wall and is greater in turbulent flow than in laminar flow. The turbulence is caused by the difference in velocity of the flow in the centre of the pipe compared to the flow near the wall. As the flow increases, the intensity of the turbulence increases and the pressure loss increases approximately to the second power.

Turbulent flows are characterised by the following properties:

Figure 8.15: Turbulence is rarely dangerous

1. pronounced self-similarity in averaging with respect to length and time, with large extension of permissible length and time scales, disordered and difficult to predict spatiotemporal structure. A cyclone is several kilometres in size, while the smallest vortices it contains are smaller than one millimetre.

2. sensitive dependence on initial conditions. Wind speed near the earth's surface varies greatly and is difficult to predict when topographical irregularities cause turbulence. Therefore, wind assessments based on local measurements are usually carried out before a wind turbine is erected.

3. sensitive dependence on boundary conditions. The turbulent flow between the jet and the wall in the wall jet tilting element passes the momentum into the slow part of the boundary layer and lets it flow away, which keeps the jet on the wall (Coanda effect).

4. sensitive dependence on boundary conditions. Snow on an airfoil dampens the large-scale turbulence in the detachment bubble and leads to flow separation even at low angles of attack. Riblets on surfaces can reduce the frictional resistance in turbulent flow, as can the small dimples on the surface of golf balls.

Turbulence can be defined as follows:

1. Randomness of the flow state: unpredictable direction and speed.

2. Diffusivity: strong and rapid mixing, in contrast to slower molecular diffusion.

3. Dissipation: kinetic energy is continuously converted into heat on all scales and divides into smaller elements in a hierarchical manner from the scales of greater expansion Turbulent flow is thus only maintained if energy is supplied from outside.

4. Non-linearity: the laminar flow becomes unstable when the non-linearities gain influence. With increasing nonlinearity, a sequence of different instabilities can occur before full turbulence develops. Emergence

Linear stability theory deals with the transformation of laminar flows into turbulent flows. For this purpose, it considers the growth of wave-shaped disturbances with small amplitude, i.e. the growth of Tollmien-Schlichting waves due to the Kelvin-Helmholtz instability.

Turbulence can also be caused by special shaping, for example in static mixers or by the small depressions called dimples on the surface of golf balls.

The harmlessness of turbulence naturally disappears when one leaves nature, quite the opposite: turbulence on the financial markets or in politics can often lead to catastrophes.

8.5 Attraction

An attractor connects chaos with order. It is a point in phase space that has an attractive effect on the system. For example, a pendulum has exactly one attractor, namely the point of rest.

The pendulum is lifted to the right or left by an angle α to a peak point. When it is released there, it moves downwards on a circular arc and becomes faster and faster. It reaches its highest speed at the zero point, then it moves upwards to an angle that is slightly smaller than α because of friction. It goes back and forth like this for a while, the angle α becomes smaller and smaller, and
finally, the pendulum comes to rest at $\alpha = 0°$. In a pendulum clock compensates
The frictional resistance is reduced by a spring that has to be wound up from time to time.

Figure 8.16: Illustration of the pendulum movement

At the maximum deflection left and right, the momentum (the product of mass and acceleration) is zero. This also applies to the maximum deflection on the right. At the point of rest, the deflection is zero, but the momentum is maximum.

A strange attractor is an attractor, i.e. a place in phase space, which represents the final state of a dynamical process whose fractal dimension is not integer and whose Kolmogorov entropy is genuinely positive. Thus, it is a fractal that cannot be described geometrically in a closed form. Sometimes the term chaotic attractor is preferred, because the "strangeness" of this object can be explained with the means of chaos theory. The dynamic process shows aperiodic behaviour.

One speaks of a strange attractor when the following conditions are fulfilled:

- Chaotic behaviour: Arbitrarily small changes of the initial state lead to completely different courses.
- Fractal structure: The attractor has a non-integer dimension.
- No splitting possibility: Each lane starting in the catchment area approaches any point of the attractor by any amount.

The properties of strange attractors can be studied quite well with some classical examples. The dynamic system used can be discrete or continuous. Continuous systems are usually described by differential equations, in phase space these systems form a line starting from the initial state, the trajectory. Due to the uniqueness of the derivative, trajectories cannot intersect at any point. From this it can be concluded that attractors in two dimensions can only have a simple structure; strange attractors and thus chaotic systems only exist in continuous dynamic systems in a phase space with at least three dimensions.

A relatively simple example of a strange attractor is the Hénon attractor (named after Michel Hénon), which is defined as a discrete system in two-dimensional space by the following equations:

$$x_{k+1} = y_k + 1 - a - x_k^2 \tag{8.5}$$

$$y_{k+1} = b - x_k \, . \tag{8.6}$$

Each mapping step can be broken down into three sub-steps: A folding-stretching operation by adding $1 - a - x_k^2$ to y. a contraction by the factor b along of the x-axis and a reflection with the x- and y-axis reversed. Choosing the parameters $a = 1, 4$ and $b = 0, 3$, the typical picture of the Hénon attractor is obtained by following the trajectory of a starting point that is sufficiently close to the attractor. Superficially, the attractor looks like a line laid in a few folds, but since it is invariant to the described fold-stretch-squeeze mirror operation, each individual line is again infinitely often divided into approximately parallel lines. A cross-section through the attractor has the structure of a Cantor set, the attractor thus has a fractal structure.

If one considers the surroundings of a point on the attractor, i.e. a circular disc with a small diameter, this is transformed by a mapping step into an elongated ellipse, which is stretched along the lines of the attractor, but has a smaller area due to the contraction step. By continuing to apply the mapping rule, the image of the point environment covers ever larger areas of the attractor, while its area approaches zero.

The De Jong attractor is defined by the following equations:

$$x_{t+1} = sin(a - y - t) - cos(b - x - t) \tag{8.7}$$

$$y_{t+1} = sin(c - x - t) - cos(d - y - t) \tag{8.8}$$

where a, b, c and d are the initial conditions.

Again, the elements that must follow these equations cannot leave a certain area.

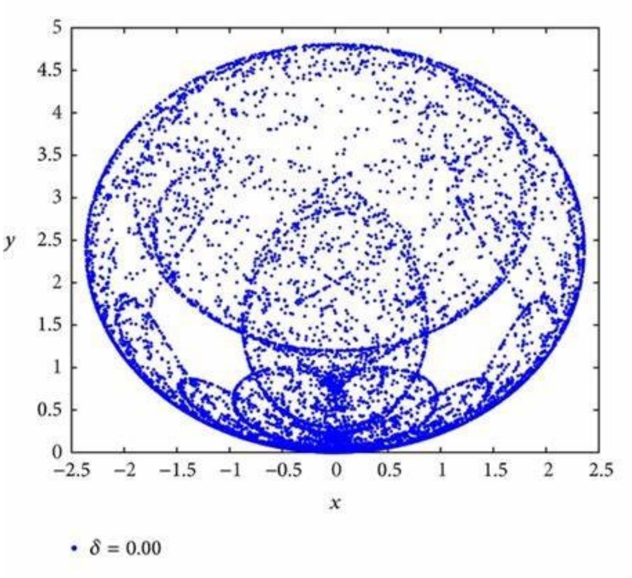


Figure 8.17: A square Hénon attractor

The Bäcker illustration describes a simple, strongly chaotic system for which all important characteristic quantities can be calculated exactly. This iterated mapping, which goes back to E. Hopf, is defined on the unit square by

$$y_{n+1} = \begin{cases} a - y_n & \text{for } 0 \leq x_n < \frac{1}{2} \\ \frac{1}{2} + a - y_n & \text{for } \frac{1}{2} \leq x_n < 1 \end{cases}$$

with $0 < a \leq \frac{1}{2}$

.

For $a = \frac{1}{2}$ the mapping is area-preserving, otherwise dissipative.

Huge attractors are hidden behind the term "black hole". A black hole is an object whose mass is concentrated in an extremely small volume and, as a result of this compactness, generates such strong gravity in its immediate vicinity that not even light can leave or pass through this area. The outer boundary of this area is called the event horizon. Nothing can cross an event horizon from the inside to the outside - no information, no radiation and certainly no matter. The fact that a "way out" is no longer even conceivable is conclusively described by the general theory of relativity through an extreme curvature of space-time.

There are different classes of black holes with their respective formation mechanisms. The easiest to understand are stellar black holes, which are formed when a star of a certain size has used up all its nuclear fuel and collapses. While the outer shells are then ejected in a supernova, the core falls to a black hole due to its gravitational pressure.

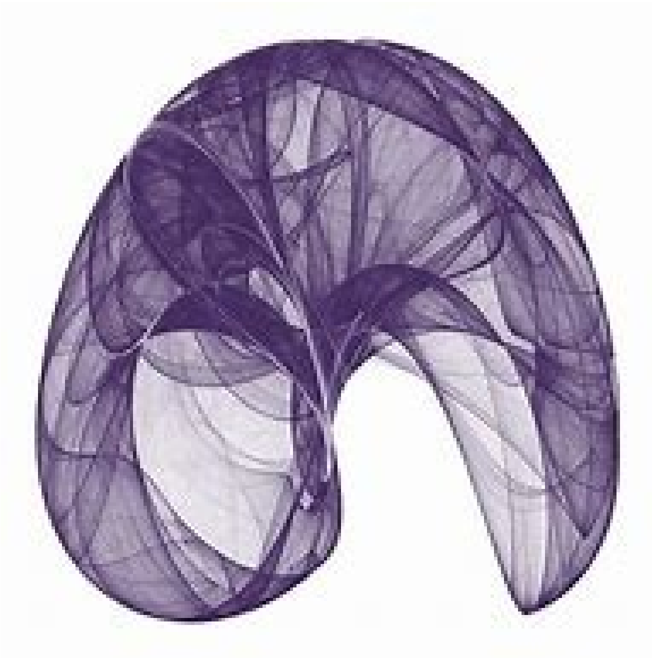


Figure 8.18: A De Jong attractor

extremely compact body together. For a hypothetical black hole of the mass of the sun, the event horizon would have a diameter of only about six kilometres, which corresponds to 230,000ths of the current diameter of the sun. At the other end of the spectrum, there are supermassive black holes of millions to billions of times the mass of the Sun, which are at the centre of galaxies and play an important role in their evolution.

Outside the event horizon, a black hole behaves like a normal mass body and can be orbited by other celestial bodies on stable paths. From the outside, the event horizon appears visually as a completely black and opaque object, in whose vicinity the space behind it is depicted as if distorted by an optical lens.

The term black hole was coined by John Archibald Wheeler in 1967. At that time, the existence of black holes, which had only been described theoretically, was considered very probable, but had not yet been confirmed by observations. Later, numerous examples of the effects of black holes were observed, e.g. the investigations of the supermassive black hole Sagittarius A* in the centre of the Milky Way in the infrared range from 1992 onwards. In 2016, the fusion of two black holes was observed by LIGO via the gravitational waves generated in the process, and in 2019, a radio telescopic image of the supermassive black hole M87* at the centre of the galaxy M87 was obtained.

A typical attraction relationship is found in the relationship between hunter and prey.

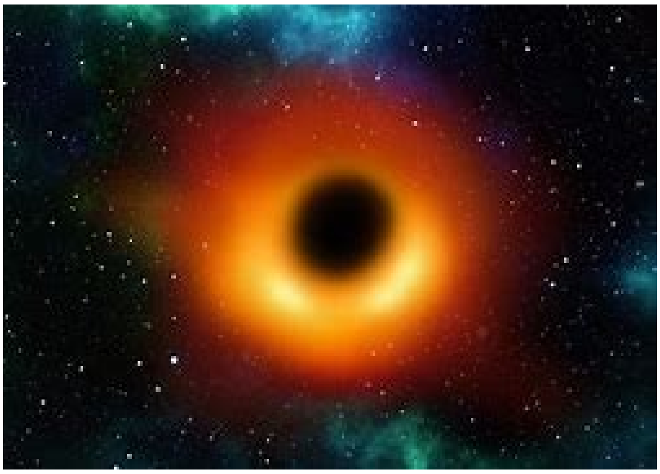


Figure 8.19: A black hole

As an example, consider the cheetah-antelope ratio. At a certain point in time, it is found that there are few cheetahs and many antelopes in a certain area. Due to the good food situation, the cheetahs multiply very quickly, their value reaches a maximum, the number of antelope a minimum. At this point, the number of cheetahs starts to decrease because there is not enough food for all of them, and the number of antelope can increase again because there are fewer enemies. The number of antelopes and cheetahs swings back and forth between a minimum and a maximum with a time lag. One can represent this relationship by two temporally offset sine oscillations.

A torus attractor is created by merging two subsystems with a total of three variables into one overall system.

- A point (of one subsystem) travels around in space on an elliptical path and another point (of the other subsystem) orbits around this point at all times.
- Through the coupling, the further point becomes the point of the total limit cycle and lies on a torus.
- Here, asymptotic predictability prevails because the system is located on the surface of the torus and not randomly in phase space.

The disruption of an orderly environment by turbulence also usually leads to chaotic conditions that cause major problems for the affected regions and people. This includes at least

- Geological turbulence (volcanic eruptions, earthquakes,...)

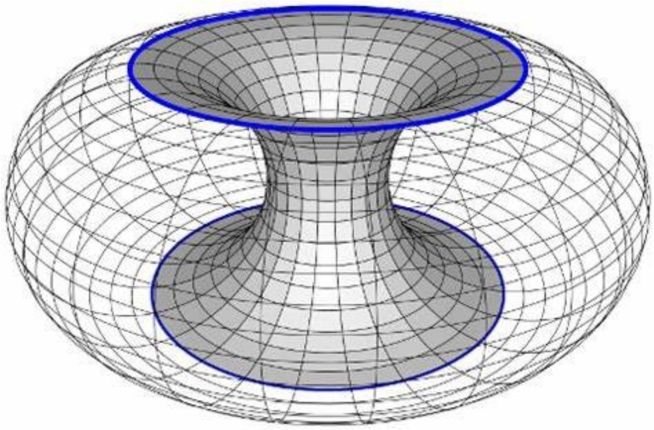


Figure 8.20: Illustration of a torus attractor

- Turbulence in the water (washing over piers and rocks, tsunamis, high and low tides, floods, ...)
- Turbulence in the atmosphere (thunderstorms, rain showers, ...)

People have been fascinated by the chaotic behaviour of turbulent currents for centuries. Leonardo da Vinci captured the phenomenon in a well-known drawing. The specific investigation came much later. In laboratory experiments, the emergence of turbulence has only been studied for a good hundred years, for example in geometrically simple environments such as pipes.

One of these experiments - a particularly famous one - can still be seen today at Manchester University: There, in 1883, Osborne Reynolds investigated when the flow of water in a straight glass tube with a circular cross-section - an idealised heating pipe - becomes turbulent. To make the flow visible, he coloured part of the water and then observed whether and when the coloured water swirled. And indeed: from a certain flow velocity, he could observe turbulence.

The study of turbulence was fruitful for mathematics: in the second half of the 20th century, it stimulated the development of the theory of dynamical systems ("chaos theory"): the researchers involved wanted to understand chaotic turbulent flows, among other things. Their mathematical investigations yielded several groundbreaking results:

The scientists were surprised to discover that in some simple model systems it is practically impossible to predict the future state - even though they knew exactly the laws according to which these systems developed over time. The systems were simply too sensitive to disturbances. One had to explain a kind of mathematical bankruptcy: The equations describing the system drifting into chaos could not be solved in general. But there was soon a glimmer of hope. Not all the mathematics of these systems was unsolvable. Special, simple solutions to the equations were found that represented typical and average states - for example, typical flow patterns. These special solutions are easier to investigate. These are mainly fixed points, i.e. unchanging solutions, or periodic orbits, where the system returns to the same state after the periodic time. These solutions and their properties reveal a lot about how exactly chaos arises, and it has even been possible to identify general mechanisms of its emergence.

Very familiar and desirable are attractors of everyday life: every supermarket, every fashion shop, every petrol station, every medical centre acts as an attractor and attracts customers or patients from a certain environment. The efforts of advertising are directed towards creating such attractors.

8.6 Bifurcation

A dynamic system can be described by a function $F(x)$ that determines the temporal evolution of the system state x. This function is now dependent on a parameter μ, which is expressed by the notation $F(x, \mu)$. Now, if the system exhibits a qualitatively different behaviour for parameter values below a certain critical value μ_c than for values above μ_c, then one speaks of the system experiencing a bifurcation in the parameter μ at μ_c. The parameter value μ_c is then called the bifurcation point.

What is a "qualitative change" can be formally described by the notion of topological equivalence or topological conjugation: as long as for two parameter values μ_1 and μ_2 the systems $F(x, \mu_1)$ and $F(x, \mu_2)$ are topologically equivalent to each other, there is no qualitative change in the above sense. The change at the bifurcation point is in most cases either a change in the number of attractors such as fixed points or periodic orbits, or a change in the stability of these objects.

Bifurcations can be graphically represented in bifurcation diagrams. For a one-dimensional system, the fixed points of the system are plotted against the parameter μ. For each parameter value, the number and position of these points is thus displayed. In addition, stable and unstable fixed points can be distinguished, e.g. by different colouring. In the case of a system with several variables one can draw similar diagrams by considering only a subspace of the phase space, for example by a Poincaré section.

The best-known bifurcation diagram is the fig tree diagram, which is derived from a logistic equation and depicts a period doubling bifurcation. It can be seen that for small parameter values there is only one stable fixed point, which at the first bifurcation point changes into an orbit of two alternating accumulation points. This orbit then doubles its period each time at further bifurcation points (i.e. it only returns to the same point after 2, 4, 8 etc. runs) until it changes into a chaotic state at a parameter value of about 3.57, where no period is recognisable at all. All these transitions can be well illustrated with the help of the bifurcation diagram.

A typical example of a bifurcation is the buckling of a rod under compressive load. Imagine a vertically standing, massless rod clamped in the ground with a load with the weight μ at the tip. The angular deviation of the rod from the vertical corresponds to the variable x.

As long as the weight remains small enough, $x = 0$ is a stable equilibrium position of the system, i.e. for small deviations, the rod automatically realigns itself to the vertical $x = 0$. If the weight μ is continuously increased, the vertical equilibrium position becomes unstable at a certain weight (the buckling load or also branching load). At the same time, two new (stable) equilibrium positions are created (for a plane system) (as the rod bends to the left or right.) The transition of the system from one (stable) to three (one unstable, two stable) equilibrium positions is the bifurcation, which in this case is a pitchfork bifurcation.

A distinction is made between the following types:

- the pitchfork bifurcation
- the saddle-node bifurcation
- the Hopf bifurcation
- the Transcritical Bifurcation
- the flip bifurcation

A very regular bifurcation occurs when you look at a person's family tree from the bottom up: each person has two parents, four grandparents, eight great-grandparents, etc. This structure quickly becomes fractal when you look at siblings, step-parents, step-siblings, and so on. In addition, the branches end at different heights because one often has no further information.

Pitchfork bifurcation, also called pitchfork or tuning fork bifurcation, is a particular type of bifurcation of a non-linear system.

The normal form of the pitchfork bifurcation is:

$$\frac{dx}{dt} = r - x + \alpha - x^3 = r - x + \alpha - x^3 \tag{8.9}$$

with $\alpha = \pm 1$, where r is the parameter to be varied for the occurrence of the bifurcation.

For $\alpha = 1$ one obtains the subcritical pitchfork bifurcation, for $\alpha = 1$ one obtains the supercritical pitchfork bifurcation

The pitchfork bifurcation has the following equilibrium points:

$$x_1^* = 0 \tag{8.10}$$

$$x_2^* = \pm\sqrt{-\frac{r}{\alpha}} \tag{8.11}$$

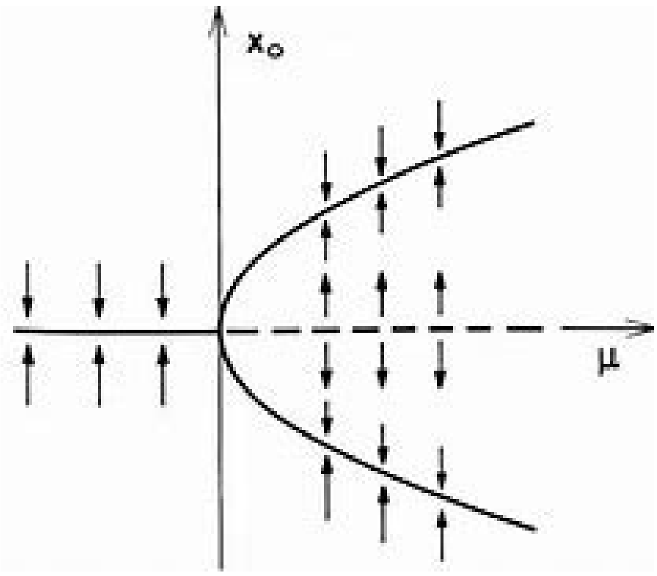


Figure 8.21: The tuning fork bifurcation

The saddle-node bifurcation is another type of bifurcation of a nonlinear dynamic system. Its normal form is

$$\frac{dx}{dt} = \mu - x(t)^2. \tag{8.12}$$

This normal form has for $\mu \geq 0$ fixed points $x_{1,2} = \pm\sqrt{\mu}$. This means that there exists for $\mu < 0$ no fixed point, for $\mu = 0$ exactly one fixed point and otherwise two. The first fixed point is stable (node), the second unstable (saddle). At the bifurcation point $\mu = 0$, the saddle and the node collide. If one considers a system with higher order in

$$\frac{dx}{dt} = \mu - x(t)^2 + O(x^3), \tag{8.13}$$

then these terms do not influence the behaviour of the system in a sufficiently small environment around the saddle-node point $\mu = 0$.

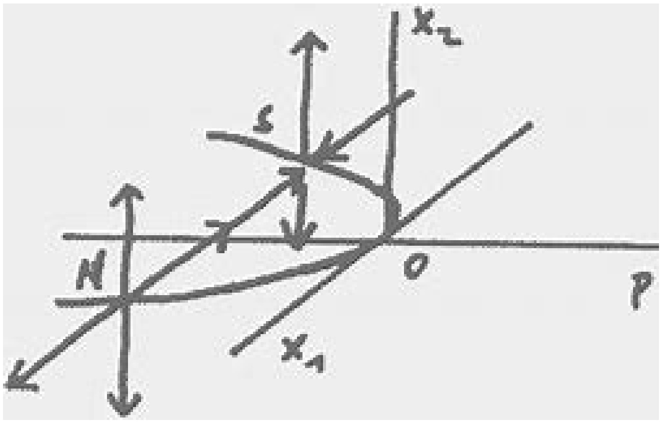


Figure 8.22: The saddle bifurcation

A Hopf bifurcation or Hopf-Andronov bifurcation is a type of local bifurcation in nonlinear systems. It is named after the German-American mathematician Eberhard Frederich Ferdinand Hopf and Alexander Alexandrovich Andronov, who treated it with Witt and Chaikin in the Soviet Union in the 1930s. The roots of the theory, however, go back to Henri Poincaré at the end of the 19th century.
19th century.

In a Hopf bifurcation, a pair of complex conjugate eigenvalues of the Jacobian matrix resulting from the linearisation of the system crosses the imaginary axis of the complex plane at an equilibrium point (fixed point) of the system; at the bifurcation point itself, the conjugate eigenvalues are therefore purely imaginary. The Hopf bifurcations can only occur in two- or higher-dimensional systems, since the linearisation of the system must have at least two eigenvalues.

The normal form of the Hopf bifurcation is

$$\frac{dz}{dt} = z((\lambda + \Im) + (\alpha + \Im\beta)|z|^2). \quad (8.14)$$

Here z is a complex quantity, t is time, the $\Im$ imaginary unit, λ, α and β are real parameters, λ is an eigenvalue.

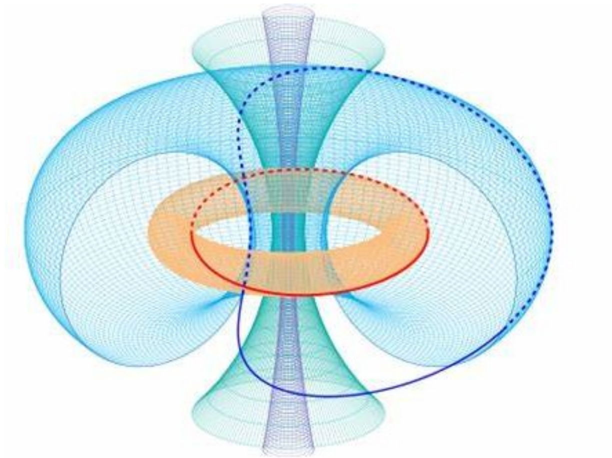


Figure 8.23: The Hopf bifurcation

Transcritical bifurcation describes a process in which the stability ("attracting" or "repelling") of two rest positions of a system is exchanged. It is thus a certain type of bifurcation of a non-linear system.

The normal form of the transcritical bifurcation is:

$$\frac{dz}{dt} = \mu - x - x^2, \quad (8.15)$$

where μ is the bifurcation parameter. The transcritical bifurcation has the following equilibrium points:

$$x_1^* = 0 \quad (8.16)$$
$$x_2^* = \mu. \quad (8.17)$$

Putting $x = (x^*_{1,2} + \delta)$ with $\delta \ll 1$ into normal form (i.e. perturbing the fixed point) and neglecting all terms of order δ^2, we get

$$\frac{d\delta}{dt} = \begin{cases} \mu\delta, & \text{at } x_1^*, \\ -\mu\delta & \text{at } x_2^*. \end{cases}$$

(8.18)

for the temporal development of the disturbance δ .

Thus, for $\mu < 0$, x_1^* is a stable fixed point (i.e. the disturbance decreases with time) and x_2^* is unstable (the disturbance grows). For $\mu > 0$ it is the other way round. At the critical value of the bifurcation parameter $\mu = 0$, the fixed point $x^* = 0$ (in this case the only fixed point) is indifferently stable.

The discrete logistic map

$$x_{n+1} = r x_n (1 - x_n) \tag{8.19}$$

also follows a transcritical bifurcation. It has the fixed points $x_1^* = 0$ and $x_2^* = 1 - \frac{1}{r}$. The origin x_1^* here is stable for $r < 1$ and unstable for $r > 1$, while x_2^* is stable for $1 < r < 3$ and loses this stability for $r > 3$.

The logistic equation can be derived from the continuous normal form by the over-gang $\frac{dx}{dt} \to x_{n+1} - x_n$ and the transformation $\frac{x_n}{1+\mu} \to x_n$,$r = 1 + \mu$ can be obtained.

9 Cellular Automata

Cellular or cellular automata are used to model spatially discrete dynamic systems, where the development of individual cells at time $t+1$ depends primarily on the cell states in a given neighbourhood and on their own state at time t. A cellular automaton is defined by the following quantities:

- a space R (cellular space)
- a finite neighbourhood N
- a set of states Q
- a local transfer function $\delta : Q^N \rightarrow Q$.

The cellular space has a certain dimensionality, it is usually one or two-dimensional, but can also be higher-dimensional. One describes the appearance of a cellular automaton by a global configuration, which is a mapping from the cellular space into the set of states, i.e. one assigns a state to each cell of the automaton. The transition of a cell from one state (local configuration) to the next is defined by state transition rules, which can be deterministic or stochastic. The state transitions occur for all cells according to the same transition function and simultaneously. Like the time steps, the cell states can be discrete. As a rule, the number of possible states is small: only a few state values are sufficient to simulate even highly complex systems.

A distinction is made between two different neighbourhoods (also called neighbourhood index):

- The Moore neighbourhood is a neighbourhood relationship in a square grid. All surfaces that have at least one corner in common with the base surface are considered neighbours. It is named after Edward F. Moore and is also referred to as a neighbourhood of 8. It corresponds to the moveable the king's possibilities in chess.

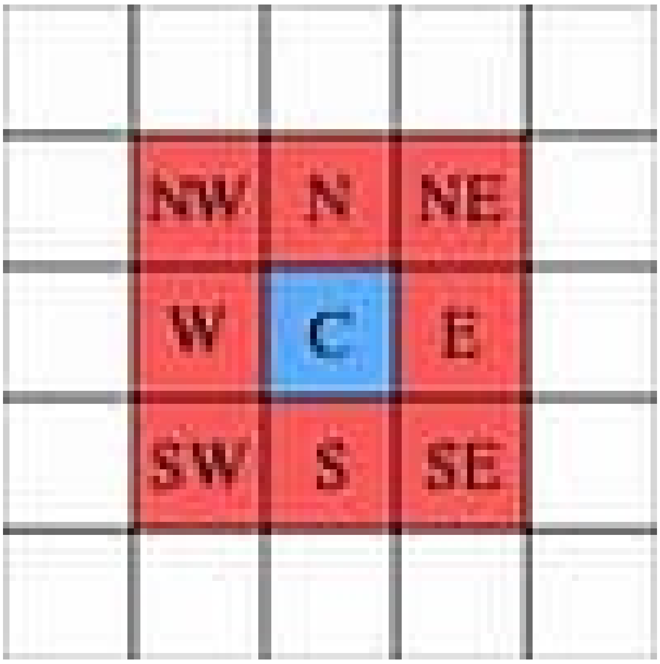


Figure 9.1: The moorland neighbourhood

- The Von Neumann neighbourhood only takes into account four cells from its neighbourhood that lie directly above and below the cell under consideration or to the left and right of it.

Many questions related to algorithms require a precise mathematical definition of an algorithm. Alan Mathison Turing (1912 - 1954) was a British logician, mathematician, cryptanalyst and computer scientist. Today he is considered one of the most influential theorists of early computer development and computer science. Turing created a large part of the theoretical foundations for modern information and computer technology. His contributions to theoretical biology also proved groundbreaking.

The computability model of the Turing machine developed by him forms one of the foundations of theoretical computer science. During the Second World War, he was instrumental in decoding German radio messages, which were encrypted with the German Enigma rotor cipher machine. Most of his work remained under lock and key even after the end of the war. The insights Turing gained in the cryptanalysis of Fish ciphers later helped in the development of the first digital, programmable electronic tube computer, ENIAC.

In 1953, Turing developed one of the first chess programs, the calculations of which he carried out himself due to a lack of hardware. The Turing Award, the most important

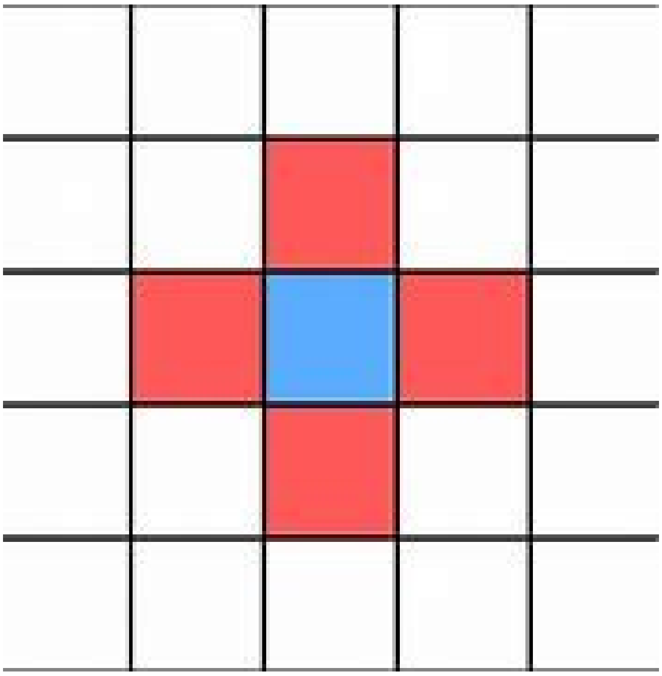


Figure 9.2: The Von Neumann neighbourhood

award in computer science, is named after him, as is the `Turing Test` to prove the existence of artificial intelligence.

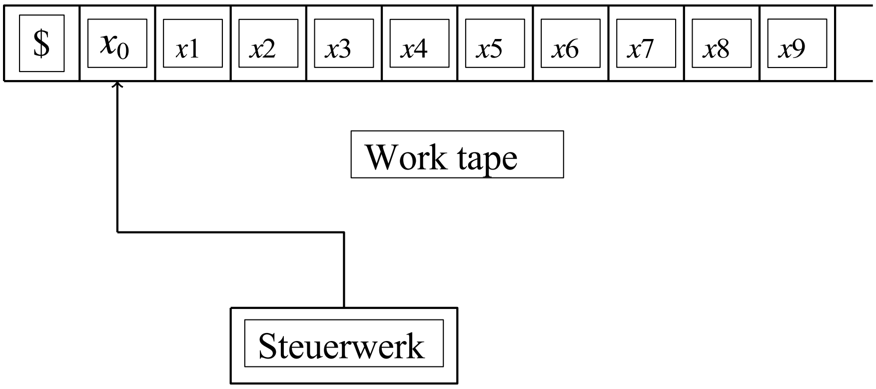


Figure 9.3: A Binding Turing Machine

At each step, the read-write head reads the current character, overwrites it with another or the same character and then moves left or right or stops. Which character is written and which movement is executed depends on the character found at the current position and on the state the Turing machine is currently in. At the beginning, the Turing machine is in a certain start state and moves to a new state at each step. The number of states the Turing machine can be in is finite. A state can be passed through several times, it says nothing about the characters present on the tape.

A Turing machine stops if no transition to a new state is defined for the current state and the read tape symbol. It therefore generally depends on the combination of state and symbol whether the Turing machine continues to calculate or stops. States in which the Turing machine stops independently of the tape symbol read are called final states. However, there are also Turing machines that never stop for certain inputs.

A Turing machine is a tuple

$$M = (Z, E, A, d, q, F)$$

with

- Z a finite, non-empty set of states of the control mechanism,
- E is the input alphabet,
- A the working alphabet with $E \subseteq A$,
- d the transition relation, $d \subseteq Z \times A \times A' \times Z$; here $A' = A \cup \{L, R\}$,
- $q \in Z$ is the start state,
- $F \subseteq Z$ the set of final states.

The concept of the single-band Turing machine is almost identical to the concept of the one-dimensional cellular automaton. A one-dimensional cellular automaton consists of a series of cells, where each cell has certain initial values and a list of rules that determine how these values are changed in each cycle. We can now assume that a whole sequence of cells are filled with the wet 0. The value 1 occurs only once. This is exactly how the initial state of the Turing machine would look. The initial situation therefore has the following shape:

. . . 001000000 . . .

As a rule, one can now determine that each cell adds its own value and the value of the left neighbour and replaces the value of the left neighbour with the new value. You start with the second cell and get $0 + 0 = 0$, then follows
$1 + 0 = 1$, , $0 + 1 = 1$. The result is

. . . 01100000 . . .

After further steps you reach

. . . 01210000 . . .
. . . 01210000 . . .
. . . 01331000 . . .
. . . 01464100 . . .
. . .

Thus, with this simple rule, one has realised an algorithm that sequentially calculates the occurring coefficients of the binomial formulae

$$(a+b)^0$$
$$(a+b)^1$$
$$(a+b)^2$$
$$(a+b)^3$$
$$(a+b)^4$$
$$\dots$$

is calculated. The algorithm works very quickly because it essentially only consists of operations that can be executed in parallel.

9.1 Conway's Game of Life

Conway's Game of Life is a game designed by mathematician John Horton Conway in 1970, based on a two-dimensional cellular automaton. It is a simple and still popular implementation of Stanisław Marcin Ulam's automaton theory.

The grid is divided into rows and columns and is ideally infinite in size. Each grid square is a cellular automaton (cell) that can assume one of two states, often referred to as alive and dead. First, an initial generation of living cells is placed on the grid. Each living or dead cell has exactly eight neighbouring cells on this playing field, which are taken into account (Moore neighbourhood). The next generation results from following simple rules.

The game can be simulated manually on a piece of paper or with computer assistance. Since a real playing field always has an edge, the behaviour must be defined there.

become. You can think of the edge as being occupied by dead cells, for example, so that some gliders change their direction of movement there. Another possibility is a torus-shaped playing field, where everything that leaves the field downwards enters again at the top and vice versa, and everything that leaves the field to the left enters again to the right and vice versa.

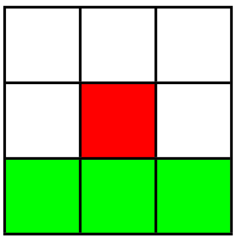
Instead of a square-gridded plane, the simulation can also be done on a hexagonal-gridded plane. Then the number of neighbouring cells is not eight, but six. There are also three-dimensional Game of Life simulations.

Another variation possibility is to introduce more than two possible states of the grid cells.

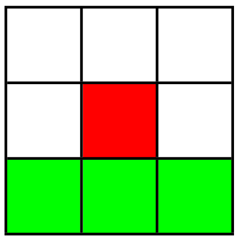
The subsequent generation is calculated for all cells simultaneously and replaces the current generation. The state of a cell (alive or dead) in the subsequent generation depends only on the current state of the cell itself and the current states of its eight neighbouring cells.

Figure 9.4 shows the rules used by Conway at the beginning.

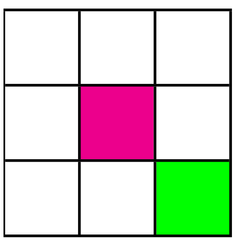
A dead cell with exactly three living neighbours is born anew in the subsequent ration.



A living cell with two or three living neighbours remains alive in the subsequent generation.



Living cells with fewer than two living neighbours die of loneliness in the subsequent generation.



Living cells with more than three living neighbours die of overpopulation in the following generation.

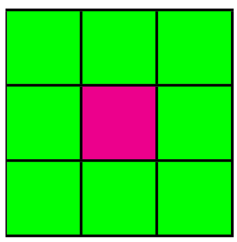


Figure 9.4: Conway's rules

With these four simple rules, a variety of complex structures emerge from certain initial patterns in the course of the game. Some remain unchanged, others oscillate and still others grow or pass away. Some structures, called gliders, move around the board. Even logical functions like AND and OR can be simulated by certain initial patterns. With this, even complex functions of circuit logic and digital computer technology can then be reproduced.

Engaging with Game of Life can be done from different perspectives, such as:

The behaviour as a whole: For some people it is interesting to see the behaviour of certain rule worlds, for example whether they explode or implode, whether they slowly shrink or whether they slowly harden.

- The biological aspect. Game of Life as a microcosm:

  For others, Game of Life is like looking into a microscope. You observe the small structures that you can count and evaluate. Here, one is particularly p l e a s e d when a new "life form" appears. Exploding, expanding or even hardening rule worlds are of no interest here.

- The economic aspect:

  Game of Life as a model of computer trading of the financial markets. According to the algorithms of computer trading, one buys a product if some, but not too many and not too few, neighbours also already own it. If too few have it, one sells it before it becomes completely worthless. If too many have it, you sell before the bubble bursts.

- The chemical aspect: energy and matter.

  If you compare the frequency and complexity of the game-of-life objects with the amount of energy and intermediate steps needed to obtain a certain chemical compound, you can set the different life objects to different energetic levels. Objects that occur in each process would then be at the level of water, carbon dioxide and sodium chloride. Objects like the balance wheel and the fountain would then be on a level like hydrochloric acid and sodium hydroxide, and objects like the gliders (LWSS, MWSS and HWSS), which can also occur by chance, would already be on the level of relatively complex compounds.

- The physical aspect: forces and initial value problem.

  Even the simplest physical laws can show any complex behaviour as a whole. Purely deterministically/mechanically, (arbitrarily) small deviations of the initial condition can lead to completely different results. Thus, an initial value problem can be formulated, which is followed by chaotic behaviour. Final states, oscillations, growth, but also permanently irregular behaviour follow.

- Game of Life as a vending machine:

  There is the type of game-of-life interested mainly in the construction of automata, i.e. such structures that work like a machine or factory. There is a federation of structures that bears a distant resemblance to an airport tarmac, where planes are constantly taking off, and in between the vehicles that maintain operations are driving to their stations.

- Game of Life as a computer model: It is possible to model a universal Turing machine and its input with the help of complex starting patterns. Conway's Game of Life is thus Turing-complete. Theoretically, any algorithmic problem that can be solved with a computer can also be calculated using Game of Life alone. This has already been shown.

- Game of Life in Theoretical Computer Science as a Decision Problem:

One can show that there is no algorithm that receives as input any two game-of-life configurations and can decide in all cases whether one configuration can arise from the other or not. This question is therefore undecidable.

On the playing field, a variety of complex structures appear with each generation step. Some typical objects can be divided into classes on the basis of any special properties they may have: they disappear, remain unchanged, change periodically (oscillate), move around the playing field, grow incessantly, and so on.

- Static objects form a class of objects that do not change during the course of the game without external influences, i.e. they represent "stable cell systems" (Fig. 9.5).

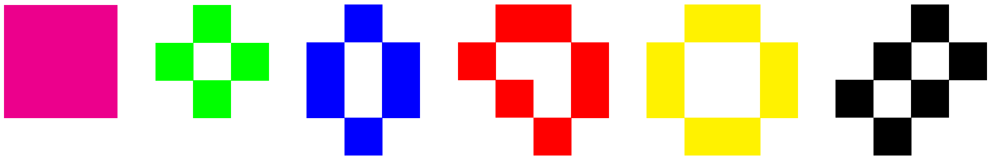


Figure 9.5: Stable cell systems

- Oscillating objects (Fig.9.6): These are objects that change periodically according to a certain pattern, i.e. they return to their initial state after a finite, fixed number of generations. The simplest cyclic configuration is a horizontal or vertical row of three living cells. In the horizontal case, a living cell is born directly above and below the cell in the middle, while the outer two cells die; thus a vertical row of three is obtained. A row of ten cells hanging together horizontally or vertically even develops into an object that has a cycle of fifteen generations, the pulsator.

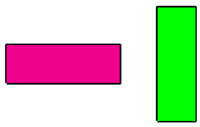


Figure 9.6: Oscillating objects

A large number of programmes can be found on the internet that follow start-up situations with different objectives over many generations. Examples include [**40**] and [**41**].

Next, an oscillating pattern is to be shown, which consists of 12 elements and has a period 15 (Fig. 9.7).

Figure 9.7: A larger oscillating object

The following constellation already reaches the initial state after the fourth step.

Each figure can be composed of five partial rectangles, the number of elements always moves from 9 to twelve and back. At each step three elements are newly created, in the next step three elements die.

The ant is a Turing machine with a two-dimensional memory and was developed by Christopher Langton in 1986. It is an example of how a

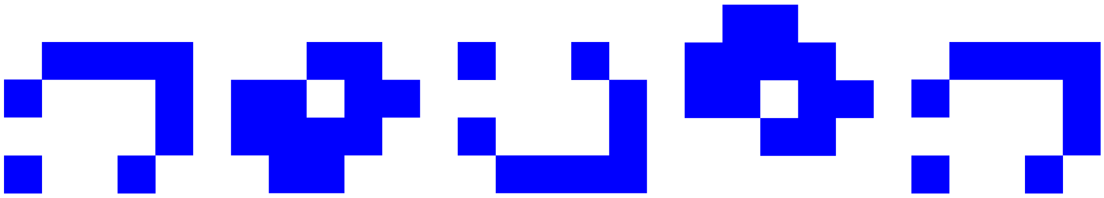


Figure 9.8: The object after four steps

deterministic (i.e. non-random) system with simple rules and a simple initial state can assume both states that appear visually surprisingly disordered to humans and states that appear regular

The ant moves in an infinite square grid of fields that can be either black or white. In the initial situation, all the squares are white and the ant sits on one of the squares and looks in a certain direction (downwards in this representation). The transition to the next state takes place according to the following rules:

- On a white field, turn 90 degrees to the right; on a black field, turn 90 degrees to the left.
- Change the colour of the field (white to black or black to white).
- Go to the next field in the current viewing direction.

In the first approx. 500 steps, symmetrical patterns occur repeatedly. After that, the ant forms a complex, chaotic pattern during about 10,000 steps. Finally, it moves on to build a regular structure ("ant trail"): It gets into the same (local) state every 104 steps; each time diagonally shifted by 2 fields, and continues building the road to infinity.

Greg Turk and Jim Propp investigated a generalisation of the classical Lang- ton ant. A field goes through a cycle of two or more field states (colours): Before moving to the next field, the ant changes the state of the current field to the next in the cycle. Each state is associated with a direction of swing, either to the left or to the right by 90 degrees. The original Langton ant is described by the rule 'RL'.

Some rules produce symmetrical mappings, others seemingly completely chaotic, although it is partly unknown whether these produce an ant trail after a sufficient number of steps.

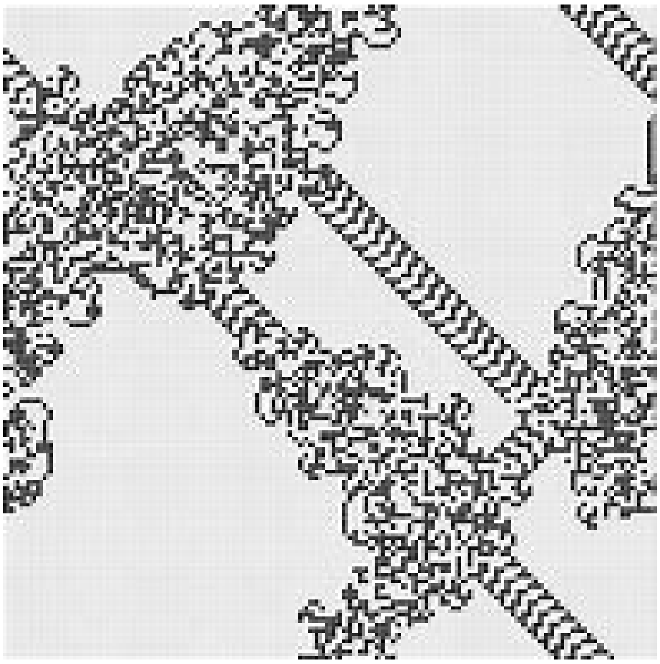


Figure 9.9: The image of an ant trail

In chaos research, solitary waves or solitons are another form of dynamic order formation in chaos. The Scottish engineer and shipbuilder John Scott Russell (1808 - 1882) had already observed a peculiar wave in the 1930s, which had seemed unthinkable to natural scientists until then. While riding along a ship canal, he noticed a wave that propagated in the canal for a long time with constant form and speed. Russell knew that waves normally dissipate quickly into chaotic turbulence due to many small disturbances.

A "soliton", on the other hand, is a wave that remains stable over time in a chaotic system. The unusual stability of solitons results from non-linear interactions in which the various oscillations in them are fed back. The oscillations in solitons therefore also exhibit a high degree of self-similarity.

The Dutch mathematicians Diederik Johannes Korteweg (1848 - 1941) and G.? de Vries (1858 - ?) developed the non-linear KdV equation (named after the first letters of their surnames) at the end of the last century, which can also be used to calculate solitons. These can only arise in a narrowly limited range of non-linear feedback: For if the wave is too strong, it soon collapses in on itself, and if it is too weak, it quickly ebbs away. Solitons can also be observed at certain river mouths, where tides regularly push tidal waves up the river. However, non-linear solitons do not only occur in narrow ship channels or river estuaries, but also in

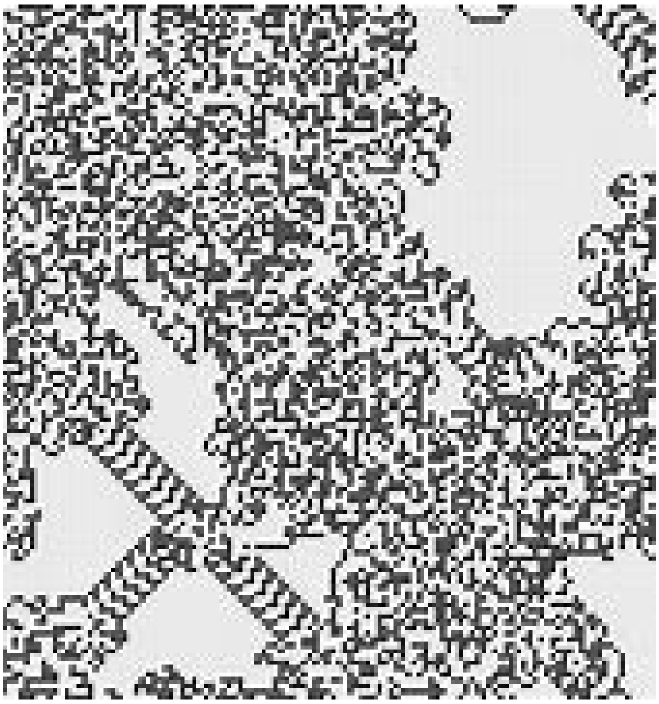


Figure 9.10: Langton's ant trail

the vastness of the oceans. This is when undersea quakes or volcanoes trigger seismic waves, and the best-known form of this is probably the tsunami that occurs in the Pacific Ocean.

This Japanese word means "great wave in the harbour", and it vividly describes the destruction that a tsunami up to 40 metres high can cause when it hits a coast. Such waves, which are stable over hundreds of kilometres, do not only occur in the Pacific Ocean, but also in other oceans; for example, the Portuguese city of Lisbon was destroyed in 1755 by an earthquake and a subsequent seismic wave. But solitons have so far not only been discovered in turbulent waters, but also in the chaotic air movements of the Earth's atmosphere. Such atmospheric solitons are created by rapid changes in air pressure, and they can also travel as stable pressure waves over hundreds of kilometres.

The resulting structures are completely unpredictable when two or more waves cross.

Non-linear systems can thus organise themselves into certain structures (such as attractors, bifurcations, intermittencies, fractals, solitons) within the framework of dynamic order formation through iteration. These discoveries originate exclusively from natural science approaches to chaos research, whereas the humanities and social sciences have been neglected until today. An essential

Figure 9.11: A single stable wave (soliton)

Figure 9.12: A tsunami hits the coastline

The reason for the neglect of chaos research in the humanities and social sciences is that chaos theory emerged from the research field of nonlinear dynamics within the framework of physics and chaos research therefore initially dealt with related disciplines such as mathematics, chemistry or biology. Even more important, however, is the reason that many chaos researchers do not even consider it possible or meaningful to apply approaches from the humanities and social sciences to chaos theory. Such reservations are also shared by the bio-physicist, chaos researcher and philosopher Bernd-Olaf Küppers, who originally pursued a career in the natural sciences and worked at the Max Planck Institute for Biophysical Chemistry in Göttingen after studying physics. There he developed a theory on the origin of biological information. Later, however, Küppers devoted himself more to the humanities aspects of his work, and he has mit-

Figure 9.13: Waves that overlap create absolute chaos

Küppers now holds a chair at the Institute of Philosophy at the University of Jena. Küppers is of the opinion that it is fundamentally unprovable whether social processes are in fact non-linear and chaotic. A presumed non-linearity cannot be definitively proven, thus violating the scientific principle of verifiability (statements are meaningless if they cannot be proven in principle). Instead, according to Küppers, they could in fact be linear in a complex way and thus calculable and predictable, even if this has only not yet been recognised. Against this possibility, however, it can again be objected that it is fundamentally irrefutable whether social processes are linear. A presumed linearity cannot be definitively disproved, which would violate the scientific principle of falsifiability (statements are meaningless if they cannot be disproved in principle). Küppers is also of the opinion that there is no scientific benefit in applying chaos theory approaches to the humanities and social sciences. The benefit of chaos research is difficult to prove even for natural sciences, such as meteorology. However, this objection is countered by the fact that, conversely, the scientific usefulness of classical, linear systems approaches is limited. Neither the weather nor social processes can be calculated or predicted, but at best statistical probabilities can be determined for limited periods of time and areas.

There are already numerous other approaches to transfer chaos theory to various areas of the humanities and social sciences. In view of Küppers' reservations, it is surprising that even he himself suggests chaos research in the humanities and social sciences for a specific field, namely for the science of history: "Apart from their enormous physical significance, chaotic systems obviously also represent an interesting model for the phenomenon of historicity. For the processes that take place in such systems are neither reversible nor repeatable. They are just as unique as all historical processes. Küppers sees the sensitive dependence of historical developments on their framework conditions as an approach for further chaos-theoretical historical research. The sociologist Walter L. Bühl has a similar approach and wants to use chaos theory to explain social change. However, he emphasises: "Chaos theory is thus primarily of importance for the description and explanation of crises and transitional states in which either previously effective attractors suddenly increase or decrease in size, shift or disappear completely, or in which they lose their previously regular structure. Chaos theory, however, is certainly useless for describing a permanent state of society or for giving a ("culturally critical") universal description of social states or developments, as is still done by the sociological classics."

In such chaos-theoretical approaches, however, it must always be borne in mind that history (like all social processes) remains fundamentally unpredictable and unpredictable because of its non-linear properties. Although historical processes are mostly determined by the deliberate actions of people, revolutions or wars cannot be predicted. Such social chaos is therefore usually seen as frightening, sinister and dangerous.

The same applies to economics, where crash situations or economic crises cannot be calculated. The non-linear interdependencies within economic processes are, however, considered to be proven, but the manifestation of chaotic behaviour is still being further investigated.

Psychology: There are also promising attempts to use chaos theory for psy- chology. For example, the psychologist Rainer Höger is pursuing an approach to explain the findings of language psychology on stuttering in terms of chaos theory.

Psychologist and physiologist Michael Stadler from the University of Bremen developed a chaos-theoretical model to explain delinquent affect behaviour. Together with Thomas Fabian from the Bremen Institute for Forensic Psychology, he states that such acts of affect usually have two special characteristics:

a) The cause of the crime is seemingly minor.

b) the scale of the response is disproportionately violent.

In such situations, socially learned behaviour patterns can fail. There are phylogenetically deeply anchored reactions such as flight, attack or stall reflex, which in such situations - in the terminology of chaos theory - represent strong attractors. Instabilities necessarily go hand in hand with fluctuations, which can be seen in bifurcation points can lead to a sliding of the behaviour into such attractors. This is precisely what can be observed in the case of affective acts. [**40]**

9.2 Three-dimensional Fractals

In earlier chapters, three-dimensional generalisations of one-dimensional or two-dimensional fractals occasionally appeared. One can look at the Menger sponge or the Sierpinski pyramid again as a reminder. They will be examined here in summary. This section is at the end because these fractals are particularly beautiful.[**43]**

An Apollonian poem is a fractal image formed from a collection of progressively smaller circles that lie within a single large circle. Each circle is tangent to the adjacent circles - in other words, the circles in t h e Apollonian poem touch at a single point. They are named after the Greek mathematician Apollonios of Perge.

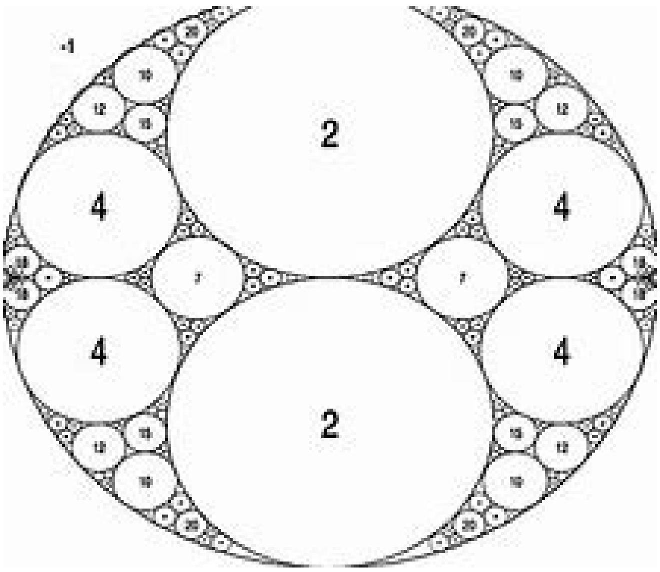


Figure 9.14: A two-dimensional apollonian figure
https://scienceblogs.de/mathlog/2014/06/20/apollonische-kreise-and-number-theory/

In 1982, algorithms were given by A. Norton that generated and displayed three-dimensional fractal structures. Quaternions were used for the first time. Quaternions are an extension of the complex numbers:

$$H = \{a + b - i + b - j + b - k, \quad i^2 = j^2 = k^2 = -1\}.$$

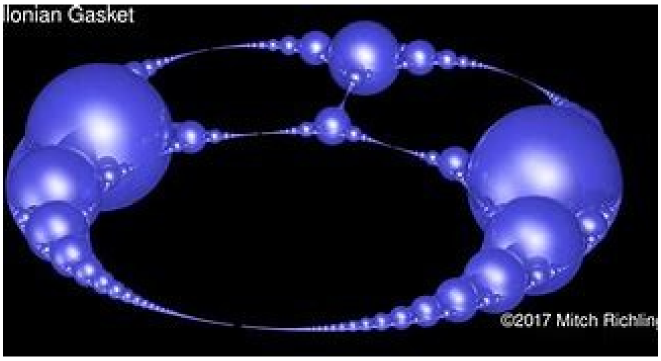


Figure 9.15: A piece of jewellery that uses the Apollonian principle three-dimensionally

Corrado Segre (1860-1924) introduces bicomplex, tri-complex, . . . , n- complex numbers. A bicomplex number is defined in the following way:

$$T = \{a + b \cdot i_1 + c \cdot i_2 + d \cdot j \mid i_1^2 = i_2^2 = -1, j^2 = 1\},$$

where

$$i_2 \cdot j = j \cdot i_2 = -i_1 , \quad i_1 \cdot j = j \cdot i_1 = -i_2 , \quad i_2 \cdot i_1 = i_1 \cdot i_2 = j$$

where a, b, c, d are real numbers. From this definition, one can derive a number of mathematical properties that can be skipped here. We continue to work with the polynomial

$$P_c(w) = w^2 + c, \qquad (9.1)$$

where $c \in T$ and $w \in T$. The polynomial thus works with bi-complex numbers. Just as before, this polynomial is repeatedly applied to itself:

$$P_c^{n+1}(w) = P_c^n(P_c(w)). \qquad (9.2)$$

The principle remains that the polynomial is applied to a bi-complex number; afterwards, the result of the first application is used as a new argument. One finds the same methodology as in the definition of the Mandelbrot set and uses bi-complex numbers instead of the previously used complex numbers. On this basis, one defines the Mandelbrot set M_2 for bi-complex numbers:

$$M_2 = \{c \in T : P_c^n(0) \text{ is finite}\}. \qquad (9.3)$$

Similarly, one generalises the Julia set to bi-complex numbers:

$$K_{2,c} = \{w \in T : P_c^n(w) \text{ is finite}\}. \qquad (9.4)$$

Now some concretisations of this general procedure will be presented. The tetrabread is defined by

$$T = \{a + b - i_1 + c - i_2\, d - j : d = 0, P_c^n\ (0) \text{ is finite}\}. \qquad (9.5)$$

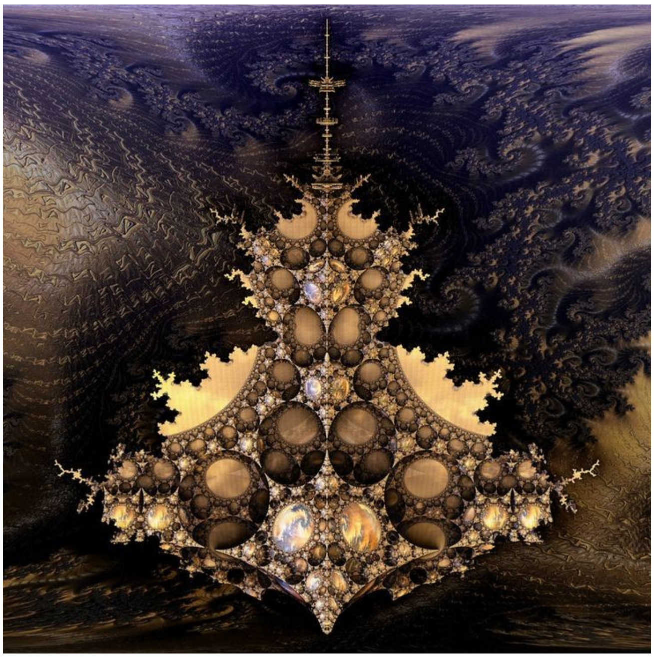


Figure 9.16: A three-dimensional Mandelbrot structure

Figure 9.17: The tetra bread

A bicomplex Julia set is described in the following way:

$$_{2,c}\, L = \{w = a + b - i_1 + c - i_2 + d_j \in T : d = 0, P_c^n \text{ is finite}\}. \qquad (9.6)$$

At http://www.3dfractals.com/ you can watch these pictures as a video with music.

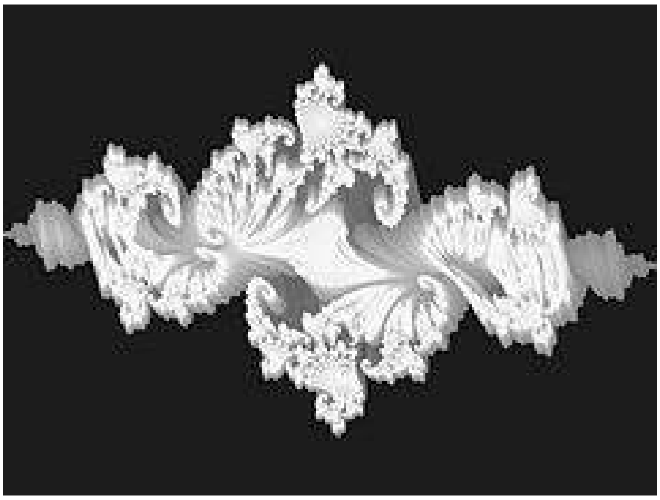


Figure 9.18: A filled Julia set